Dynamische Portfoliooptimierung mit partieller Information und Expertenmeinungen

Ralf Wunderlich

BTU Cottbus–Senftenberg

Gemeinsam mit with R. Frey (WU Wien)
J. Sass, D. Westphal (TU Kaiserslautern)
A. Gabih (Marrakech)
S. Schütze, H. Kondakji (BTU)

Workshop Mathematik in Forschung und Lehre , Pobershau, 22.-23. September 2016

Aufgabenstellung

Finanzmarkt	mit riskanten und risikolosen Wertpapieren zeitstetig, kontinuierliches Handeln
Startkapital	$x_0 > 0$
Anlagezeitraum	$[0, T]$
Ziel	maximaler mittlerer Nutzen des Endvermögens
Aufgabe	Bestimmung einer optimalen Anlagestrategie Wie viele Anteile von welchem Wertpapier sind wann zu halten **?**
Besonderheiten	Drift abhängig von unbeobachtbaren Faktorprozess Investor beobachtet nur Wertpapierpreise und Expertenmeinungen

Klassisches Black-Scholes Modell

Bond $dB_t = rB_t\,dt, \quad B_0 = 1 \quad \Longrightarrow$ Bondpreis $B_t = e^{rt}$

r risikoloser stetiger Zinssatz

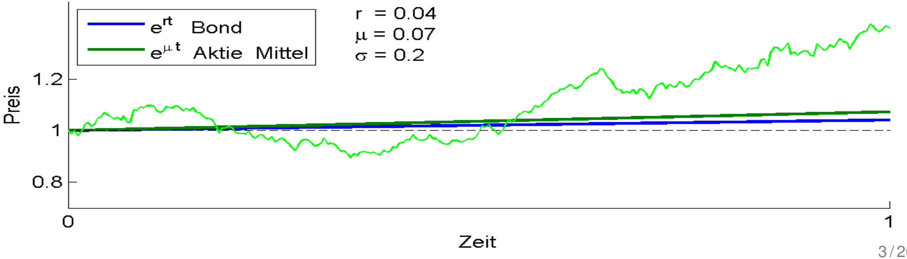
Aktie Wahrscheinlichkeitsraum $(\Omega, \mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}, P)$

Aktienpreis $S_t = \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right), \quad t \in [0, T]$

Renditen (returns) $dR_t = \frac{dS_t}{S_t} = \mu\,dt + \sigma\,dW_t$

W_t Wiener-Prozess,

μ mittlere Aktienrendite, Drift; σ Volatilität



Portfolio

Anfangskapital $X_0 = x_0 > 0$

Vermögen zur Zeit t investiert in $X_t = X_t(\underbrace{\pi_t^0}_{\text{Bond}} + \underbrace{\pi_t^1}_{\text{Aktie 1}} + \ldots + \underbrace{\pi_t^n}_{\text{Aktie n}})$

π_t^k Wertanteil von Aktie k, $k = 1, \ldots, n$

Strategie $\boldsymbol{\pi_t = (\pi_t^1, \ldots, \pi_t^n)^\top}$

Selbstfinanzierungsbedingung und Annahme $r = 0$ $\Rightarrow$

Vermögensgleichung

$X_t = X_t^\pi$ genügt linearer stochastischer DGL mit Anfangswert x_0

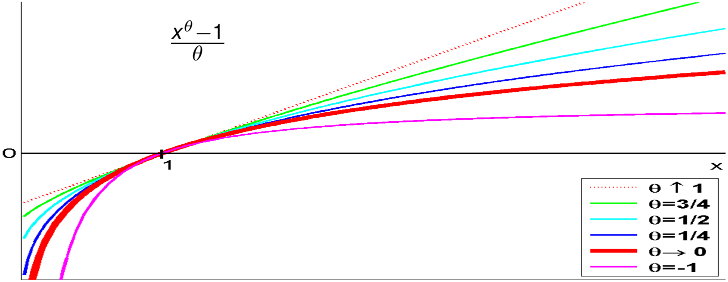
$$\begin{aligned} dX_t^\pi &= X_t^\pi \boldsymbol{\pi_t^\top} (\mu \, dt + \sigma dW_t) \\ X_0^\pi &= x_0 \end{aligned}$$

Nutzenfunktion

$U : [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ streng monoton wachsend, konkav

$U(x)$ bewertet „Nutzen“ des Endvermögens X_T

$$U(x) = \begin{cases} \frac{x^\theta}{\theta} & \text{for } \theta \in (-\infty, 1) \setminus \{0\} \quad \text{power utility} \\ \log x & \text{for } \theta = 0 \quad \text{log-utility} \end{cases}$$



Grenzfälle $U(x) - \frac{1}{\theta} \xrightarrow[\theta \to 0]{} \log x$, $U(x) \xrightarrow[\theta \uparrow 1]{} x$

Klassische Optimierungsaufgabe

Vermögensgleichung $dX_t^\pi = X_t^\pi \pi_t^\top (\mu \, dt + \sigma \, dW_t), \quad X_0^\pi = x_0$

Strategie $\pi = (\pi_t)_{t \in [0,T]}$

Zulässige Strategien $\mathcal{A}(x_0) = \{(\pi_t) :$ Integrabilitätsbedingungen, $X_t^\pi \geq 0, \; \forall t \in [0, T] \}$

Optimierungsaufgabe

$$\max_{\pi \in \mathcal{A}(x_0)} \quad E\left[U(X_T^\pi)\right]$$

Lösung optimaler Wertanteil $\pi_t^* = \dfrac{1}{1-\theta} (\sigma\sigma^\top)^{-1} \mu = \text{const}$

Merton (1969,1973)

Nobelpreis für Wirtschaftswissenschaften 1997

Modelle mit partieller Information über die Drift

- Drift $\boldsymbol{\mu}$ ist sehr schwierig zu schätzen

  benötigen Beobachtungen über lange Zeiträume

  ist nicht konstant

  abhängig vom Zustand der Ökonomie

- Idee: μ ist abhängig von Zeit und von weiterer "Rauschquelle"

  ist nicht $\mathbb{F}^R$–adaptiert (nicht beobachtbar)

  $\mathbb{F}^R = (\mathcal{F}_t^R)_{t\in[0,T]} \subset \mathbb{F}$ von Renditen R generierte Filtration

  Investor kann nur Renditen R_t (bzw. Aktienpreise S_t) beobachten,

  jedoch nicht Drift μ_t und Wiener–Prozess W_t

  $\Longrightarrow$ Modell mit partieller Information

- Problem "Lernen" der Drift aus beobachteten Aktienpreisen

  Schätzung bzw. Filter für μ_t

Financial Market with Partial Information

$(\Omega, \mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}, P)$ filtered probability space

Money market with interest rate 0

Stock market prices $S_t = (S_t^1, \ldots, S_t^n)^\top$, returns $dR_t^i = dS_t^i / S_t^i$

$$dR_t = \mu_t \, dt + \sigma \, dW_t^R$$

$$\begin{aligned} W^R &= (W_t^R)_{t\in[0,T]} && n\text{-dimensional Brownian motion} \\ \mu &= (\mu_t)_{t\in[0,T]} && \text{stochastic drift, independent on } W^R \\ \sigma & && \text{volatility, non-singular} \end{aligned}$$

Information investor filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]} \subset \mathbb{G}$

classical situation $\mathcal{F}_t = \mathcal{F}_t^R = \mathcal{F}_t^S \subset \mathcal{F}_t^{W^R,\mu} \subset \mathcal{G}_t$

filtering with observation R and signal μ

we may also consider additional information leading to $\mathcal{F}_t^H \subset \mathcal{G}$

optimal strategies depend on filter $\widehat{\mu}_t^H = E[\mu_t | \mathcal{F}_t^H]$
and its dynamics and distribution

Trading and Portfolio Optimization

- X_t wealth (portfolio value) at time t
- $\pi = (\pi_t)_{t \in [0,T]}$ trading strategy

  π_t^i is fraction of wealth X_t invested in stock i

  π has to be $\mathbb{F}^H$-adapted
- Wealth $X_t = X_t^\pi$ is controlled by π and satisfies

$$dX_t = X_t \pi_t^\top dR_t = X_t \pi_t^\top (\mu_t dt + \sigma dW_t^R), \quad X_0 = x_0$$

- Evaluation of terminal wealth with utility function U, e.g.

$$U_\theta(x) = \frac{x^\theta}{\theta}, \quad \theta < 1, \ \theta \neq 0, \qquad \text{or} \qquad U_0(x) = \log(x)$$

- Stochastic control problem: maximize expected utility

$$E[U(X_T^\pi)] \quad \text{over admissible strategies } \pi \quad \text{for } x_0 > 0$$

Optimal Strategies in Special Cases

- For constant μ: $\pi_t^* = \frac{1}{1-\theta}(\sigma\sigma^\top)^{-1}\mu = \text{const}$ Merton strategy
- For stochastic μ, information $\mathbb{F}^H$ and $U = U_0 = \log$
  optimal strategy is obtained by substituting filter $\widehat{\mu}_t^H$ for μ
  (Certainty equivalence principle)
  - Proof: (for $n = 1$ and $x_0 = 1$):
    $$\log X_T^\pi = \int_0^T \Big(\pi_t\,\mu_t - \frac{1}{2}(\sigma\pi_t)^2\Big)dt + \int_0^T \pi_t\sigma dW_t^R$$
  - For $\mathbb{F}^H$-adapted π we obtain
    $$E[\log X_T^\pi] = \int_0^T E\left[\pi_t\,E\left[\mu_t \mid \mathcal{F}_t^H\right] - \frac{1}{2}(\sigma\pi_t)^2\right] dt + 0$$
    $$= \int_0^T E\left[\pi_t\,\widehat{\mu}_t^H - \frac{1}{2}(\sigma\pi_t)^2\right] dt$$
  - Pointwise maximization yields $\pi_t^* = \sigma^{-2}\widehat{\mu}_t^H$
- In general we expect a dependency of π_t^* on filter $\widehat{\mu}_t^H$ and its dynamics

Drift Models

- Bayesian Model:

  μ is random but time-independent

  KARATZAS/XUE (1991)

- Linear Gaussian Model (LGM) or Kim-Omberg model

  μ is an Ornstein-Uhlenbeck process

  leads to Kalman filter

  LAKNER (1998), BRENDLE (2006)

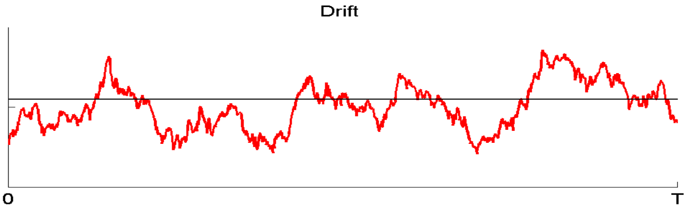
- Hidden Markov Model (HMM)

  μ is a continuous-time Markov chain

  leads to Wonham or HMM filter

  SASS, HAUSSMANN (2004), RIEDER, BÄUERLE (2005)

Linear Gaussian Model (LGM)



Drift is a Gaussian mean-reversion (Ornstein-Uhlenbeck) process

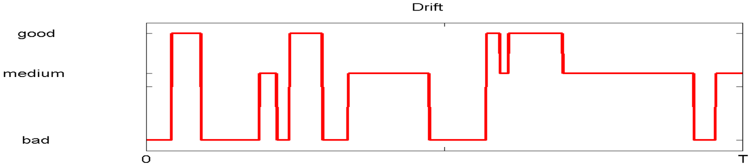
$$d\mu_t = \kappa(\overline{\mu} - \mu_t)dt + \delta dW_t^{\mu}, \qquad \mu_0 \sim \mathcal{N}(m_0, \eta_0)$$

where W_t^{μ} is a Brownian motion (in)dependent of W_t^R

Closed-form solution available

Stationary distribution for $t \to \infty$ is $\mathcal{N}(\overline{\mu}, \frac{\delta^2}{2\kappa})$

Hidden Markov Model (HMM)



Drift $\mu_t = \mu(Y_t)$ is a finite-state Markov chain, independent of W_t^R

Y has state space $\{e_1, \dots, e_d\}$, unit vectors in $\mathbb{R}^d$

$\mu(Y_t) = MY_t$ where $M = (\mu_1, \dots, \mu_d)$ contains states of drift

switching properties defined by generator or rate matrix $Q \in \mathbb{R}^{d \times d}$

diagonal: $Q_{kk} = -\lambda_k$ exponential rate of leaving state k

conditional transition prob. $P(Y_t = e_l \mid Y_{t-} = k, Y_t \neq Y_{t-}) = Q_{kl}/\lambda_k$

initial distribution $(\rho^1, \dots, \rho^d)^\top$

Einbeziehung von Zusatzinformationen

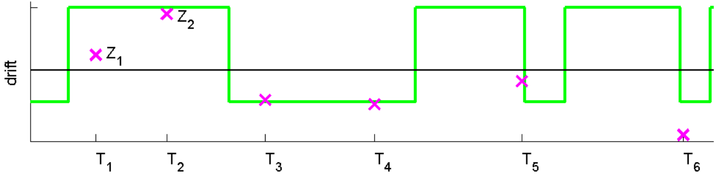
- Driftschätzungen (Filter) aus Beobachtungen von Aktienpreisen sind oft sehr ungenau.
  Das Rauschen (Volatilität) dominiert das Signal (Drift).
- Praktiker verwenden zur Portfoliooptimierung oft Markowitz-Modell, ein (sehr einfaches statisches) Ein-Perioden-Modell.

  In einer Erweiterung, dem Black-Litterman Modell, kombinieren sie empirische Driftschätzungen aus Marktdaten mit Zusatzinfos, den „Experten-Meinungen“ :
  - eigene Einschätzung der Performance z.B.
    „Aktie A wächst um 5%“
    „Rendite von Aktie B ist 3% größer als die von Aktie C“
  - Empfehlungen von Analysten, Ratings, Unternehmensberichte
- Ziele — Übertragung dieses Ansatzes auf dynamische Modelle

  Lösung von dynamischen Portfoliooptimierungs-Problemen

Experten-Meinungen

Modellierung durch markierten Punktprozess $I = (T_n, Z_n)$



- Zu Zeitpunkten T_n beobachtet Investor ZG $Z_n \in \mathcal{Z}$, $n = 1, 2, \ldots$
- T_n: Sprungzeitpunkte eines Poisson-Prozesses mit Intensität $\lambda > 0$
- Z_n abhängig von aktuellen Zustand Y_{T_n}, Dichte $f(Y_{T_n}, z)$

Beispiele

- Absolute Vorhersage:

$$Z_n = \mu(Y_{T_n}) + \sigma_\varepsilon \varepsilon_n, \quad (\varepsilon_n) \quad \text{i.i.d. } \mathcal{N}(0,1)$$

  Vorhersage "S wächst um 5%" $\longrightarrow$ $Z_n = 0.05$

  σ_ε beschreibt Vertrauen/Genauigkeit der Vorhersage

- Relative Vorhersage (2 Aktien):

$$Z_n = \mu_1(Y_{T_n}) - \mu_2(Y_{T_n}) + \sigma_\varepsilon \varepsilon_n$$

Maximizing Log-Utility in a Model with Gaussian Drift

- We consider one stock (n = 1) with

$$\begin{aligned} \text{returns} \quad dR_t &= \mu_t dt + \sigma dW_t^R \\ \text{and} \quad \text{drift} \quad d\mu_t &= \kappa(\overline{\mu} - \mu_t)dt + \delta dW_t^\mu, \quad \mu_0 \sim \mathcal{N}(m_0, \eta_0) \end{aligned}$$

- N expert opinions arrive at fixed times $0 = t_0 < t_1 < \ldots < t_{N-1} < T$.
  Views are modeled as Gaussian unbiased estimates Z_k of the current drift.

$$Z_k = \mu_{t_k} + \sqrt{\Gamma_k}\varepsilon_k, \quad \text{for i.i.d. } \varepsilon_1, \ldots, \varepsilon_N \sim \mathcal{N}(0,1).$$

  $\Gamma_k > 0$ describes the confidence of the expert.

- We distinguish four information regimes for the investor

$\mathbb{F}^R$	observing	returns only
$\mathbb{F}^E$		expert opinions only
$\mathbb{F}^C$		both returns and expert opinions
$\mathbb{F}^F$	having	full information

- We have to compute filters $\quad \widehat{\mu}_t^H = E[\mu_t | \mathcal{F}_t^H]$
  and conditional variances $\quad q_t^H = E[(\mu_t - \widehat{\mu}_t^H)^2 | \mathcal{F}_t^H] \quad$ for $H = R, E, C, F$.

Filtering: Returns Only $(H = R)$

- For $H = R$, we are in the classical Kalman filter case and get

$$d\widehat{\mu}_t^R = \kappa(\overline{\mu} - \widehat{\mu}_t^R)dt + \sigma^{-2} q_t^R \left(dR_t - \widehat{\mu}_t^R dt\right), \quad \widehat{\mu}_0^R = m_0,$$

  and deterministic conditional variance satisfying the Riccati equation

$$\frac{d}{dt} q_t^R = \delta^2 - 2\kappa q_t^R - \sigma^{-2} (q_t^R)^2, \quad q_0^R = \eta_0.$$

  For $n = 1$ we have a closed-form solution for q_t^R.
- For $t \to \infty$ we have $q_t^R \to q_\infty^R := \kappa\sigma^2 \left(\sqrt{1 + (\frac{\delta}{\kappa\sigma})^2} - 1\right)$
- q_t^R is decreasing if $\eta_0 > q_\infty^R$ and
  increasing if $\eta_0 < q_\infty^R$.

Filtering: Returns and Expert Opinions $(H = C)$

- Between the information dates the filter and the conditional variance evolve as in regime $H = R$ (Kalman filter).
- At the information dates t_k the expert opinion $Z_k \sim \mathcal{N}(\mu_{t_k}, \Gamma_k)$ leads to a Bayesian update

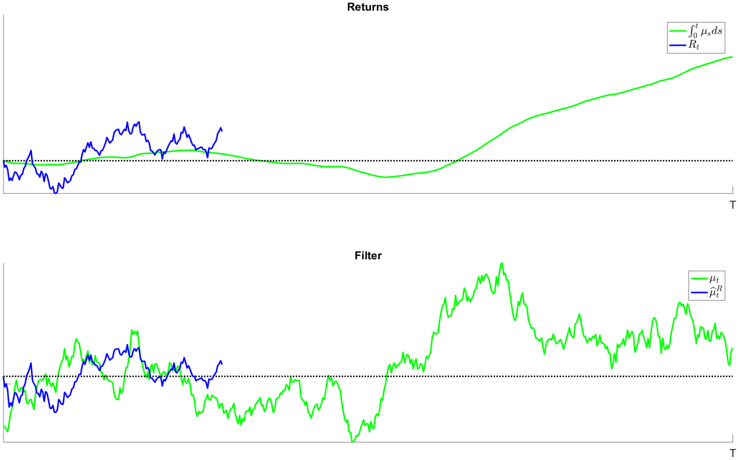
$$\begin{aligned} \widehat{\mu}^C_{t_k} &= \rho_k \, \widehat{\mu}^C_{t_k-} + (1-\rho_k)\, Z_k \\ q^C_{t_k} &= \rho_k q^C_{t_k-} \end{aligned}$$

with the factor

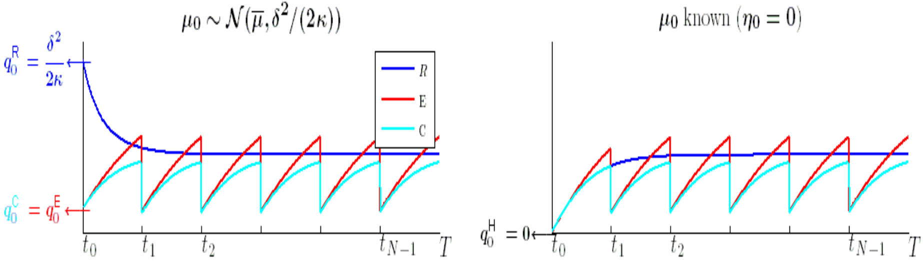
$$\rho_k = \frac{\Gamma_k}{\Gamma_k + q_{t_k-}} \in (0,1)$$

- For $H = E$ we get corresponding updating formulas and between the information dates we can consider the limiting case $\sigma = \infty$ in regime $H = C$.
- For $H = F$ we have full information and thus $\widehat{\mu}^F_t = \mu_t$ and $q^F_t = 0$.

Example: Filter $\widehat{\mu}_t^H$



Example: Conditional Variance q_t^H



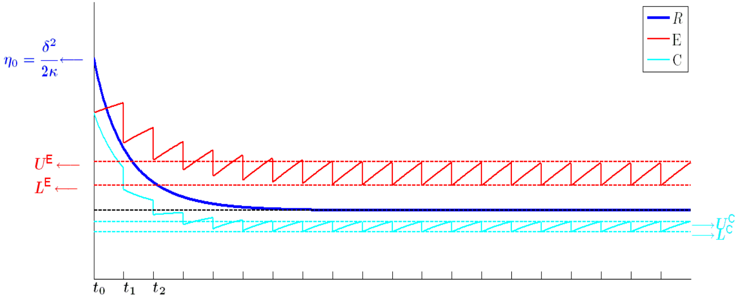
It holds $\quad q_t^C \leq \min\{q_t^R, q_t^E\}$

Asymptotics of Conditional Variance for $t \to \infty$

- Let $T = \infty$
- $q_t^R \to q_\infty^R$ for $t \to \infty$
- For equidistant expert opinions with $\Gamma_k = \Gamma > 0$ we get for $H = E, C$

$$\limsup_{t\to\infty} q_t^H = U^H \quad \text{and} \quad \liminf_{t\to\infty} q_t^H = L^H,$$

where $U^H > L^H > 0$ can be computed explicitly.



Diffusion Approximation for High-Frequency Experts

- Expert opinions arrive more frequent and become "less confident":

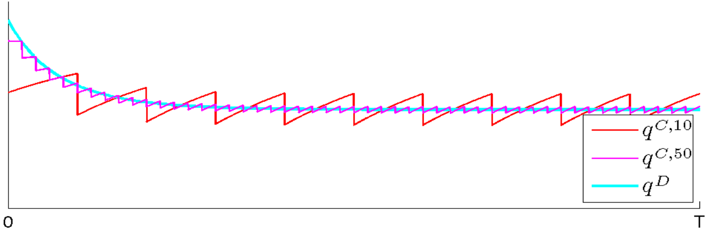
  Let $t_k = k\Delta_n$ with $\Delta_n = T/n$

  $\Gamma_k = \Gamma = \sigma_J^2/\Delta_n = n\sigma_J^2/T$ with $\sigma_J > 0$

- Introduce "continuous-time expert" as diffusion process $dJ_t = \mu_t dt + \sigma_J dW_t^J$
- Regime $H = D$: observe R and J, Kalman filter $\widehat{\mu}^D$ and cond. variance q^D
- For $n \to \infty$ it holds

$$|q^{C,n}(t) - q^D(t)| \to 0 \quad \text{uniformly for all } t \in [0, T]$$

$$\int_0^T E[\, |\widehat{\mu}_t^{C,n} - \widehat{\mu}_t^D|^2 \,]dt \to 0$$



Optimal Expected Logarithmic Utility

$$V(x_0) = \sup_{\pi \in \mathcal{A}^H} E[\log X_T^*]$$

Already known: for $U = U_0 = \log$ and information $\mathbb{F}^H$ the optimal strategy is

$$\pi_t^* = \sigma^{-2} \widehat{\mu}_t^H$$

Therefore, $$V^H(x_0) = E[\log X_T^{\pi^*}] = \log x_0 + E\Big[\int_0^T \Big(\pi_t^* \, \widehat{\mu}_t^H - \frac{1}{2} (\sigma \pi_t^*)^2 \Big) dt \Big]$$

$$= \log x_0 + \frac{1}{2\sigma^2} \int_0^T \underbrace{E\big[(\widehat{\mu}_t^H)^2\big]}_{E[\mu_t^2] - q_t^H} \, dt$$

Theorem (Gabih/Kondakji/Sass/W. 2014)

$$V^H(x_0) = \log x_0 + \frac{1}{2\,\sigma^2} \Big(\int_0^T E[\mu_t^2] \, dt - \int_0^T q_t^H \, dt \Big)$$

where the integrals (in all four cases) can be computed explicitly.

Properties derived for q_t^H allow to derive corresponding properties for $V^H(x_0)$.

Efficiency

We want to quantify the monetary value of the information.

Investor	F	$H \ (= R, E, C)$
Information (observations)	$\mathbb{F}^F$	$\mathbb{F}^H$
Initial capital	$x_0^F = 1$	x_0^H
Opt. terminal wealth	X_T^F	X_T^H

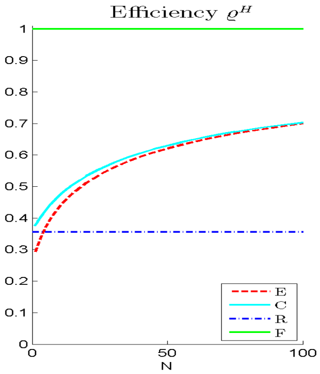
How much initial capital x_0^H needs H to obtain the same expected utility as F?

$$\text{Solve} \quad V^F(1) = V^H(x_0^H):$$
$$x_0^H = \exp\left(\frac{1}{2\sigma^2}\int_0^T q_t^H \, dt\right)$$

Loss of information $\quad x_0^H - 1$

Efficiency $\quad \varrho^H = 1/x_0^H$

for (non-fully informed) H-investor .



Efficiency: Numerical Example

Efficiency ϱ^H in % for various numbers N

	ϱ^H	
R	35.63	
N	E	C
10	43.49	47.12
100	69.94	70.36
1.000	88.37	88.39
10.000	96.09	96.09
100.000	98.74	98.74
1.000.000	99.60	99.60
10.000.000	99.87	99.87
F	100.00	

Extension to models with $n \geq 1$ assets : SASS, WESTPHAL, W. (2016)

Power Utility

- Formulierung als stochastisches Kontrollproblem
  Lösung mit Methoden des Dynamic Programming
  Analysis und Numerik der Hamilton-Jacobi-Bellman-Gleichung (HJB)

- Für zufällige Informationszeitpunkte $T_1, T_2, \ldots$ ist
  HJB-Gleichung eine partielle Integro-DGL (PIDE)
  erfordert numerische Lösung

- Für feste Informationszeitpunkte $t_0, \ldots, t_{N-1}$
  Dynamic Programming Principle liefert eine Rückwärtsrekursion

- HMM $\rightarrow$ Dissertation Stephan Schütze (2016)
  LGM $\rightarrow$ Dissertation Hakam Kondakji (2017+x)