On mean square solutions of random differential equations in scales of BANACH spaces

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Introduction

- Many real systems are described using ordinary or partial differential equations.
- There exists a well developed mathematical theory, especially regarding existence and uniqueness theorems for solutions.
- One method which is broadly used to incorporate uncertainty in mathematical models consists in the usage of stochastic models and hence the investigation of random or stochastic ordinary or partial differential equations.
- For stochastic (ordinary or partial) differential equations existence and uniqueness theorems for solutions are also an important field of investigation (Itô calculus).
- The interest for existence and uniqueness theorems for solutions is much weaker with respect to random differential equations.
- Here: some remarks to mean square solutions for scalar random ordinary differential equations (RODE).

Some historical remarks

- Mathematical investigation was done mainly around the sixties.
- Relevant books are e.g.
  - Srinivasan/Vasudevan, Introduction to random differential equations and their applications, 1971.
  - Bharucha-Reid, Random integral equations, 1972.
  - Bunke, Ordinary differential equations with random parameters („Gewöhnliche Differentialgleichungen mit zufälligen Parametern“), 1972 (in German).
  - Soong, Random differential equations in science and engineering, 1973.
  - Kirillov, Random equations, 1982 (in Russian).
  - Bobrowski, Introduction to random ordinary differential equations, 1987 (in Polish).
  - Sobczyk, Stochastic differential equations with applications to physics and engineering, 1990.
  - Neckel/Rupp, Random differential equations in scientific computing, 2013.
  - Han/Kloeden, Random ordinary differential equations and their numerical solution, 2017.

Deterministic ODEs

- Here for simplicity: equations for real-valued functions $x(t), t \in \mathbb{T}$, with some time interval $\mathbb{T} \neq \emptyset$.
- **Assumption 1**
  Consider the explicit differential equation of first order
  $$\dot{x} = f(x, t)$$
  with given function $f : \mathbb{R}^2 \supseteq D \to \mathbb{R}$, $\emptyset \neq D$ open
  and with given initial condition $x(t_0) = x_0$ with $(x_0, t_0) \in D$.
- **Definition 1**
  A solution is a differentiable function $\varphi : \mathbb{T} \to \mathbb{R}$, such that
  $\varphi(t_0) = x_0$ and $\forall\, t \in \mathbb{T} : (\varphi(t), t) \in D$ and $\dfrac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = f(\varphi(t), t)$.
- Main existence theorems are theorems by:
  - (Cauchy-)Peano
  - Picard-Lindelöf
  - Carathéodory

(Cauchy-)Peano theorem

- **Theorem 1**
  If Assumption 1 is fulfilled and the function f is continuous in a neighborhood of (x_0, t_0), **then** there exists at least one solution of the initial value problem on an open interval containing t_0. If f is continuous on D, each solution can be continued up to the boundary of D.
- In general uniqueness of solutions to the initial value problem cannot be guaranteed.
  Example: $\dot{x} = 3x^{2/3}\,,\ x(0) = 0\,; \quad \varphi_1(t) = 0\,,\ \varphi_2(t) = t^3\,.$
- The differential equation is equivalent to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)\,\mathrm{d}s$$

  (for the corresponding existence interval) under the assumptions of the (Cauchy-)Peano Theorem.

PICARD-LINDELÖF theorem

- **Theorem 2**
  If Assumption 1 is fulfilled and the function f is continuous and LIPSCHITZ continuous with respect to the variable x in a neighborhood $\mathcal{U}(x_0, t_0)$ of (x_0, t_0), i.e., it holds $\exists\, L > 0$

  $$\forall\,(x_1, t), (x_2, t) \in \mathcal{U}(x_0, t_0) \quad |f(x_1, t) - f(x_2, t)| < L|x_1 - x_2|\,,$$

  then there exists a unique solution of the initial value problem on an open interval containing t_0. If f is continuous and locally LIPSCHITZ continuous on D, each solution can be continued up to the boundary of D.
- Proof for example via successive approximations

$$\varphi_0(t) = x_0 + \int_{t_0}^{t} f(x_0, s)\,\mathrm{d}s$$

$$\varphi_k(t) = x_0 + \int_{t_0}^{t} f(\varphi_{k-1}(s), s)\,\mathrm{d}s\,, \quad k = 1, 2, 3, \ldots .$$

CARATHÉODORY condition

- **Definition 2**
A solution in the extended sense or in the sense of CARATHÉODORY is an absolutely continuous function $\varphi : \mathbb{T} \to \mathbb{R}$, such that $\varphi(t_0) = x_0$, $\forall\, t \in \mathbb{T} : (\varphi(t), t) \in D$ and $\dfrac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = f(\varphi(t), t)$ for all $t \in T$ except on a set of LEBESGUE measure zero.
- **Remark** A solution in the sense of CARATHÉODORY is a continuous solution of the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)\,\mathrm{d}s.$$

(The function f need not to be continuous, the integral is understood as LEBESGUE integral.)
- **Definition 3**
The function $f : D \ni (x, t) \mapsto f(x, t) \in \mathbb{R}$ is said to satisfy the CARATHÉODORY condition, if it is LEBESGUE measurable in t for fixed x and continuous in x for (almost) all fixed t.

CARATHÉODORY theorem

- **Theorem 3**
  Let Assumption 1 be fulfilled and let the function f satisfy the CARATHÉODORY condition in a neighborhood $\mathcal{U}(x_0, t_0)$ of (x_0, t_0).

  If there exists a locally LEBESGUE integrable function $m > 0$ with $|f(x,t)| \leq m(t)$ on $\mathcal{U}(x_0, t_0)$,
  then there exists a local solution in the sense of CARATHÉODORY.

  If there additionally exists a locally LEBESGUE integrable function $\ell > 0$ with

  $$\forall\, (x_1, t), (x_2, t) \in \mathcal{U}(x_0, t_0) \quad |f(x_1, t) - f(x_2, t)| < \ell(t)|x_1 - x_2|\,,$$

  then the solution in the sense of CARATHÉODORY is unique.

Random ODEs

- All stochastic quantities are defined on a fixed probability space $(\Omega, \mathcal{F}, \mathrm{P})$.
- An initial value problem for a random explicit ordinary differential equation of first order can be written formally as

$$\dot{X}(t,\omega) = F(X(t,\omega), t, \omega)\,,\ X(t_0,\omega) = X_0(\omega)\,,\ \omega \in \Omega\,. \quad \text{(RODE)}$$

- Different concepts for derivatives of random functions and other issues lead to different solution concepts.
- Basic theoretical issues are the measurability assumption for time values of random functions and a time interval of existence independent of $\omega \in \Omega$.
- Mostly random differential equations arise by substituting deterministic parameters with random functions. (This can be questionable from a serious modeling point of view.)

Pathwise solutions

- **Definition 4**
  A random function $(X(t); t \in \mathbb{T})$ is called a pathwise solution to (RODE), if for all fixed $\omega_0 \in \Omega_1 \in \mathcal{F}$ with $\mathrm{P}(\Omega_1) = 1$ the deterministic functions $\mathbb{T} \ni t \mapsto X(t, \omega_0) \in \mathbb{R}$ are solutions to the corresponding deterministic initial value problem
  $$\dot{X}(t, \omega_0) = F(X(t, \omega_0), t, \omega_0)\,,\ X(t_0, \omega_0) = X_0(\omega_0)\,,\ \omega_0 \in \Omega_1\,.$$
- **Theorem 4**
  If almost surely the conditions of the existence theorem for deterministic ODEs are fulfilled and the solutions define a random function defined on a fixed intervall, **then** there exists a pathwise solution to the initial value problem for the random ordinary differential equation.
- **Remark** In many applications this kind of solution concept is used. The derivative is then also the derivative in the sense of almost sure convergence.

Mean-square solutions

- Often one is interested in first and second order moments of a solution.
- Then under mild assumptions the derivative is in the sense of mean-square convergence, i.e., the derivative of the abstract function $\mathbb{T} \ni t \mapsto X(t) \in \mathrm{L}^2(\Omega, \mathcal{F}, \mathrm{P}) =: \mathrm{L}^2$, i.e.,

$$\lim_{h \to 0} \mathsf{E}\left[\left| \frac{X(t+h) - X(t)}{h} - \dot{X}(t) \right|^2 \right] = 0$$

- Therefore it seems to make sense to consider such random ordinary differential equations as differential equation in a HILBERT space (or more generally in a BANACH space.
- Doing so the modeling is changed somehow: now realizations are not necessarily differentiable, the derivative is defined by "averaging over all possible realizations".

(Cauchy-)Peano theorem in Banach spaces

- There is no simple generalization of the (Cauchy-)Peano theorem for abstract functions with values in a Banach space.
- One can state generalizations with stronger assumptions.
- **Theorem 5** (Godunov)
  For every infinite-dimensional Banach space B, $t_0 \in \mathbb{R}$, $x_0 \in \mathrm{B}$, there exists a continuous mapping $f : \mathrm{B} \times \mathbb{R} \to \mathrm{B}$, such that there exists no solution to the initial value problem for an abstract function $x : \mathbb{T} \to \mathrm{B}$

$$\dot{x} = f(x,t), \quad x(t_0) = x_0 .$$

PICARD-LINDELÖF theorem in BANACH spaces

- **Theorem 6**
  Let be given $t_0 \in \mathbb{R}$, $x_0 \in \mathrm{B}$ (a BANACH space) and an abstract function $\mathrm{B} \times \mathbb{R} \supset D \ni (x,t) \mapsto f(x,t) \in \mathrm{B}$, which is defined, continuous as well as bounded and LIPSCHITZ continuous with respect to x in a neighborhood $\mathcal{U}$ of (x_0, t_0).
  Then on an open interval $\mathbb{T}$ containing t_0 there exists an abstract function $x : \mathbb{T} \to \mathrm{B}$, which is continuously differentiable (with respect to the norm) and which satisfies on $\mathbb{T}$ the equation

$$\dot{x}(t) = f(x(t), t) \quad \text{and also} \quad x(t_0) = x_0 .$$

- **Remark** Also in this case the initial value problem is equivalent to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s) \,\mathrm{d}s$$

(with a BANACH space valued RIEMANN (or BOCHNER integral).

Special case: linear equation with random coefficient

- Consider the random initial value problem on $\mathbb{R}$
$$\dot{x} = \kappa x \,, \qquad x(0) = 1 \,,$$
with a random variable κ.
- The unique pathwise solution is $x(t) = \exp(\kappa t) \,, t \in \mathbb{R} \,.$
- If $\mathrm{E}[\exp(2\kappa t)] < \infty$ it is also the mean-square solution.
- If κ is unbounded, for every $0 \neq x_0 \in \mathrm{L}^2$ the function $f(x,t) = \kappa x$ cannot be defined on a L^2 neighborhood of x_0 as an operator into L^2, so that the PICARD-LINDELÖF theorem in BANACH spaces cannot be used in this case. and in more complicated situations.
- This holds also in more complicated situations (nonlinear equations) and other spaces $\mathrm{L}^p(\Omega, \mathcal{F}, \mathrm{P})$ with $p \geq 1 \,.$
- So corresponding statements in the literature are complicated and not so applicable and useful.

Observation

- Assume $\alpha > 0$, $p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$. Then HÖLDER's inequality gives
$$(\mathrm{E}[|\kappa\xi|^{\alpha}])^{1/\alpha} \leq (\mathrm{E}[|\kappa|^{\alpha q}])^{1/(\alpha q)} (\mathrm{E}[|\xi|^{\alpha p}])^{1/(\alpha p)} .$$
- Hence if $\forall \gamma > 0 : \mathrm{E}[|\kappa|^{\gamma}] < \infty$, then from $\mathrm{E}\left[|\xi|^{\beta}\right] < \infty$ with $\beta > 1$ it follows $\mathrm{E}[|\kappa\xi|^{\alpha}] < \infty$ for all $0 < \alpha < \beta$, i.e., the multiplication operator (function) maps from $\mathrm{L}^{\beta}(\Omega, \mathcal{F}, \mathrm{P})$ to $\mathrm{L}^{\alpha}(\Omega, \mathcal{F}, \mathrm{P})$.
- So one can try to consider the differential equation in a whole family of BANACH spaces. A similar situation exist in the functional analytic treatment of certain partial differential equations (evolution equations) by considering such equations in a scale of BANACH spaces.

Scale of BANACH spaces

- **Definition 5**
  A family $(\mathrm{B}_\alpha)_{\alpha_0 \leq \alpha \leq \beta_0}$ of BANACH spaces is called a scale of BANACH spaces, if
  (i) for $\beta > \alpha$ the space B_β is densely embedded in the space B_α and hence with $0 < C(\alpha, \beta) < \infty$
  $$\forall x \in \mathrm{B}_\beta : \ \|x\|_{\mathrm{B}_\alpha} \leq C(\alpha, \beta) \|x\|_{\mathrm{B}_\beta} ;$$
  (ii) and there exists a function $0 < C(\alpha, \beta, \gamma) < \infty$, such that for $\alpha_0 \leq \alpha < \beta < \gamma \leq \beta_0$ it holds
  $$\forall x \in \mathrm{B}_\gamma : \ \|x\|_{\mathrm{B}_\beta} \leq C(\alpha, \beta, \gamma) \|x\|_{\mathrm{B}_\alpha}^{\frac{\gamma-\beta}{\gamma-\alpha}} \ \|x\|_{\mathrm{B}_\gamma}^{\frac{\beta-\alpha}{\gamma-\alpha}}$$

  The scale of BANACH spaces is called a normal scale, if one can take $C(\alpha, \beta) = C(\alpha, \beta, \gamma) = 1$.
- **Remark** The family of spaces $(\mathrm{L}^\alpha(\Omega, \mathcal{F}, \mathrm{P}))_{\alpha \geq 2}$ constitutes a normal scale of BANACH spaces.

PICARD-LINDELÖF theorem in a normal scale of BANACH spaces

- **Theorem 6** (NISHIDA, see e.g. also TUTSCHKE)
  Let be given a normal scale of BANACH spaces $(\mathrm{B}_\alpha)_{\alpha_0<\alpha<\beta_0}$
  Assume further
  - $\exists\, R > 0,\, T > 0\ \forall\, \alpha_0 < \alpha < \beta < \beta_0$ the function $(x,t) \mapsto f(x,t)$ maps continuously $\mathcal{U}_{\beta;R}(x_0) \times [0,T]$ with $\mathcal{U}_{\beta;R}(x_0) := \{x \in B_\beta \,:\, \|x - x_0\|_{\mathrm{B}_\beta} \le R\}$ into B_α ;
  - $\exists\, K > 0\ \forall\, t \in [0,T]\,, \forall\, \alpha_0 < \alpha < \beta_0 \,:\, \|f(x_0,t)\|_{\mathrm{B}_\alpha} \le \dfrac{K}{\beta_0 - \alpha}$;
  - $\exists\, C > 0\ \forall\, t \in [0,T]\,, \forall\, \alpha_0 < \alpha < \gamma < \beta_0$

    $\forall\, x_1, x_2 \in \mathcal{U}_{\gamma;R}(x_0) \,:\, \|f(x_1,t) - f(x_2,t)\|_{\mathrm{B}_\alpha} \le \dfrac{C\|x_1 - x_2\|_{\mathrm{B}_\gamma}}{\gamma - \alpha}$.

  Then $\forall\, \alpha_0 < \alpha < \beta_0$ there exists in B_α an unique solution to

  $$\dot{x}(t) = f(x(t),t)\,, \qquad x(t_0) = x_0\,.$$

  on some time interval $[0;T_1]$ with $T_1 \le T$.

Application to random ODEs

- This result can be applied to random ordinary differential equations, for linear equations as well as for nonlinear equations, where for the right hand side a Lipschitz condition with random coefficients with finite moments of all orders or finite moment generating function can be shown.

Selected further literature

- H. Gajewski & K. Kröger & K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Berlin, 1974.
- P. Hájek & M. Johanis, *On Peano's theorem in Banach spaces.* J. Differential Equations, 249 (2010), pp. 3342-3351.
- S.G. Krejn & Yu.I. Petunin & E.M. Semenov, *Interpolation of linear operators (in Russian)*, Moscow, 1978.
- T. Nishida, *A note on a theorem of Nirenberg.* J. Differential Geometry, 12 (1977), pp. 629-633.
- W. Tutschke, *Solution of initial value problems in classes of generalized analytic functions*, Leipzig, 1989.
- W. Walter, *Gewöhnliche Differentialgleichungen. Eine Einführung*, Berlin, 2000.