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On a stochastic arc furnace model

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Workshop Mathematik in Forschung und Lehre,
Saalfelder Höhe

- Based on the instantaneous power balance (Acha et al (1990)) the nonlinear ordinary differential equation

$$k_1 r^n(t) + k_2 r(t)\frac{dr(t)}{dt} = \frac{k_3}{r^{m+2(t)}} i^2(t) \qquad \text{(EAF)}$$

describes the dynamics of the arc radius $r(t)$ with time t and
  - $i(t)$ the arc current;
  - $k_1, k_2, k_3 > 0$ model coefficients determined experimentally;
  - $m,\ n \in \{0, 1, 2\}$ parameters, reflecting different working conditions of the furnace.
- The arc voltage is represented by $u(t) = \frac{k_3}{r^{m+2}(t)} i(t)$.
- Assumption $n = 2$ and substitution $y = r^{m+4} \Longrightarrow$ (EAF) results in

$$\frac{dy}{dt} = -\beta y(t) + f(t)$$

where $f(t) := \frac{(m+4)k_3}{k_2} i^2(t)$, $\beta := \frac{(m+4)k_1}{k_2} > 0$.

A stochastic model:

- One proposal: Replace the deterministic parameter k_3 by

$$\left(k_3\,(1+X_t)^2\right)_{t\in\mathbb{R}}$$

  with a mean square continuous second order stochastic process $(X_t)_{t\in\mathbb{R}}$ with continuous paths.
- Random ordinary differential equation:

$$\begin{aligned}\frac{\mathrm{d}Y_t}{\mathrm{d}t} &= -\beta Y_t + F_t\\ F_t &= f(t)(1+X_t)^2 = c_f i^2(t)\,(1+X_t)^2\\ i(t) &= a\sin(\omega t)\end{aligned}$$

- With pathwise and mean square random solution $(Y_t)_{t\in[t_0,\infty]}$:

$$Y_t = y_0 e^{-\beta(t-t_0)} + \int_{t_0}^{t} e^{-\beta(t-s)} F_s\,\mathrm{d}s, \qquad Y_{t_0} = y_0$$

Stationary excitation and solution

- Assume $(X_t)_{t\in\mathbb{R}}$ as mean square continuous strong stationary stochastic process with continuous paths
  $\Rightarrow \ \left((1+X_t)^2\right)_{t\in\mathbb{R}}$ is a stationary process.
- $\Longrightarrow$ Periodically correlated solution on $\mathbb{R}$

$$Y_t^{\text{per}} = \int_0^\infty \mathrm{e}^{-\beta s} F_{t-s}\,\mathrm{d}s = \int_0^\infty \mathrm{e}^{-\beta s} f(t-s)(1+X_{t-s})^2\,\mathrm{d}s$$

(steady state solution, limit for $t_0 \to -\infty$).

- Mean value for periodically correlated solution

$$\begin{aligned}\mathrm{E}\left[Y_t^{\text{per}}\right] &= y_{\text{per}}(t)\,\mathrm{E}\left[(1+X_0)^2\right] \\ &= c_1\left[1 - c_2\sin(2\omega t+\psi)\right]\mathrm{E}\left[(1+X_0)^2\right].\end{aligned}$$

Stationary excitation and solution

- For covariance function $\mathrm{Cov}_{Y^{\mathrm{per}}}(s,t) = \mathrm{Cov}\left[Y_s^{\mathrm{per}}, Y_t^{\mathrm{per}}\right]$ an integral formula can be derived.
- Moment functions for random arc radius $(R_t)_{t\in[0,T]}$ or random voltage $(U_t)_{t\in[0,T]}$ cannot be calculated without further assumptions.
- **Reason**: nonlinear relationship between the functions R and Y, and respectively U, Y and X. So it holds

$$R_t = Y_t^{\frac{1}{m+4}}, \qquad U_t = k_3\,(1+X_t)^2\,Y_t^{-\frac{m+2}{m+4}} i(t).$$

Stationary Ornstein-Uhlenbeck process (O-U process)

- $(X_t)_{t\in\mathbb{R}}$ stationary Gaussian process with continuous paths and $\mathrm{E}[X(t)] = 0$, $\mathrm{Cov}[X_t, X_t] = \frac{\sigma^2}{2\theta}\exp(-\theta|t-s|)$ with $\theta > 0$, $\sigma > 0$, $t, s \in \mathbb{R}$.
- Can for $t \geq 0$ be considered as solution of stochastic differential equation

$$\mathrm{d}X_t = -\theta X_t \mathrm{d}t + \sigma \mathrm{d}W_t$$

with standard Wiener process $W = (W_t\,;\, t \geq 0)$ and initial value $X_0 \sim N\left(0, \frac{\sigma^2}{2\theta}\right)$ independent of W.

$\Rightarrow$ Possibility of investigating the coupled system of nonlinear stochastic differential equations

$$\mathrm{d}\begin{pmatrix} Y_t \\ X_t \end{pmatrix} = \begin{pmatrix} -\beta Y_t + f(t)\,(1+X_t)^2 \\ -\theta X_t \end{pmatrix} \mathrm{d}t + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} \mathrm{d}W_t.$$

Mean function of u with Monte-Carlo method

- Simulating 1000 paths $(x_t)_{t\in[0,T]}$ of the Ornstein-Uhlenbeck process (here with $\theta = 1.3$ and $\sigma = 6$).
- Numerical computation of

$$y_t = y_0 e^{-\beta t} + \int_0^t e^{-\beta(t-s)} f(s)(1+x_s)^2 \, \mathrm{d}s, \qquad t \in [0,T]$$

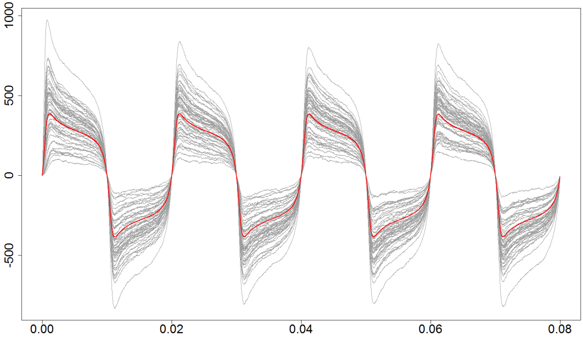
  as the solution of the initial value problem of the ordinary differential equation.
- Computation of arc voltage paths with the formula

$$u_t = \sqrt{2} k_3 (1+x_t)^2 |I| y_t^{-\ell} \sin(\omega t), \quad \ell = \frac{m+2}{m+4}, \quad t \in [0,T]$$

Mean function of u with Monte-Carlo method

- First 50 simulations of u (grey lines) and estimated mean function (red line) with parameters

t_0	y_0	$\|I\|$	ω	m	k_1	k_2	k_3	θ	σ
0	0.01	50	100π	0	3 000	1	12.5	1.3	6



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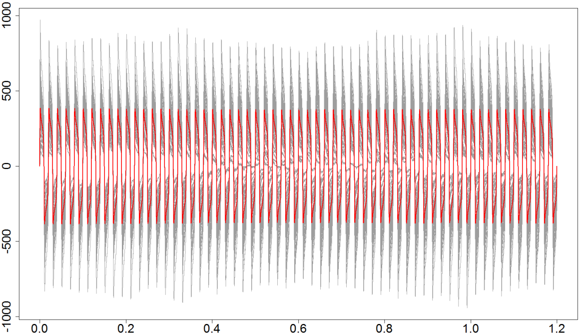
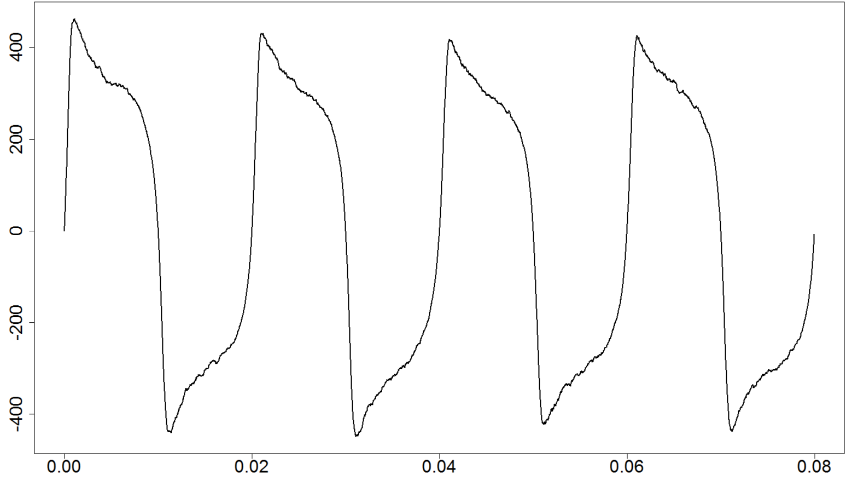


- One single path of u:



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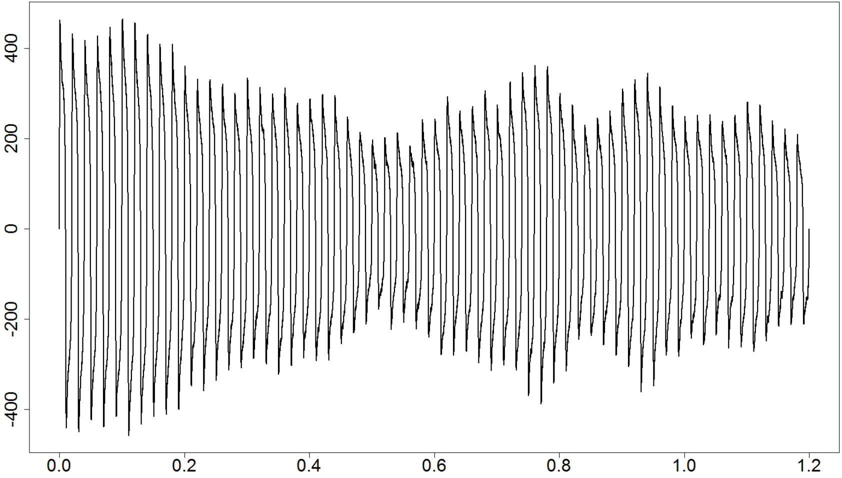


Karhunen-Loève expansion of O-U process

- On time interval $[0, T > 0]$ a.s and in mean square

$$X_t = \sum_{k \in \mathbb{N}} \sqrt{\lambda_k} \xi_k \phi_k(t), \qquad\qquad t \in [0, T],$$

with i.i.d. N(0,1) random variables $\xi_k = \frac{1}{\sqrt{\lambda_k}} \int_0^T X_t \phi_k(t)\,\mathrm{d}t$.

- Covariance operator $C_X : \mathsf{L}^2([0,T]) \to \mathsf{L}^2([0,T])$ is defined by

$$(C_X g)(s) := \int_0^T \mathrm{Cov}[X_t, X_s]\, g(t)\,\mathrm{d}t = \int_0^T \frac{\sigma^2}{2\theta} e^{-\theta|s-t|} g(t)\,\mathrm{d}t.$$

- $\lambda_k > 0, \phi_k(\cdot)$ eigenvalues and corresponding orthonormalized (in $\mathrm{L}^2[0,T]$) eigenfunctions of the covariance operator:

$$C_X \phi_k = \lambda_k \phi_k \qquad \text{for } k \in \mathbb{N}.$$

Karhunen-Loève expansion of O-U process

- Eigenvalues can be calculated numerically (solutions of transcendental equations), eigenfunctions for given eigenvalue can be calculated analytically (eg. cf. Corlay, S.; Pages, G. (2010)).
- For numerical computations we need to truncate the Karhunen-Loève expansion:

$$X_t^N = \sum_{k=1}^{N} \sqrt{\lambda_k} \xi_k \phi_k(t), \qquad N \in \mathbb{N}.$$

- Y^N pathwise solution, where X is replaced by X^N:

$$Y_t^N = y_0 e^{-\beta(t-t_0)} + \int_{t_0}^{t} e^{-\beta(t-s)} f(s) \left(1 + X_s^N\right)^2 \mathrm{d}s.$$

Polynomial chaos expansions of Y

- Can be represented as polynomial chaos expansion:

$$Y_t^N = \sum_{k=0}^{M} y_k(t)\Psi_k,$$

with $M = \frac{(N+2)(N+1)}{2} - 1$. $(\Psi_k)_{k\in\mathbb{N}_0}$ are orthogonal random variables with properties:

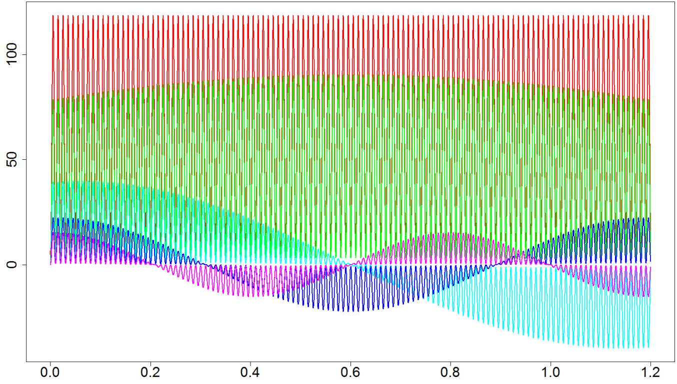
  - For every Ψ_k exists a N-variate polynomial p_k, such that

$$\Psi_k = p_k\left(\xi_1, \xi_2, \ldots, \xi_N\right).$$

  - $\Psi_0 = 1$ and $\Psi_k = \xi_k$ for $k \in \{1, \ldots, N\}$.
  - To consider are only the multivariate Hermite polynomials up to degree 2.

Numerical computations

- Selected parameters: $N = 20 \Longrightarrow M = \frac{(N+2)(N+1)}{2} - 1 = 230$, $T = 1.2$
- y_0 in red, y_1 in green, y_2 in light blue, y_3 in blue, y_4 in purple:



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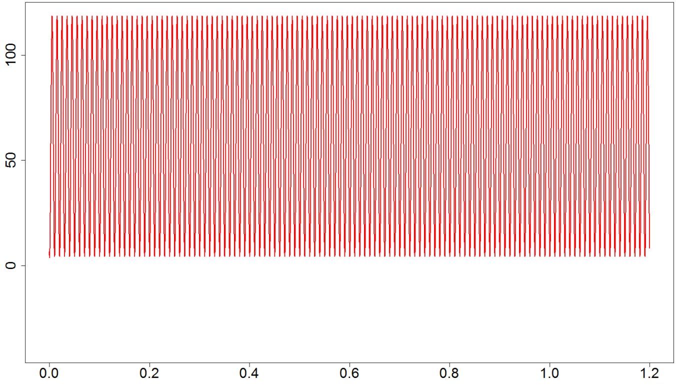

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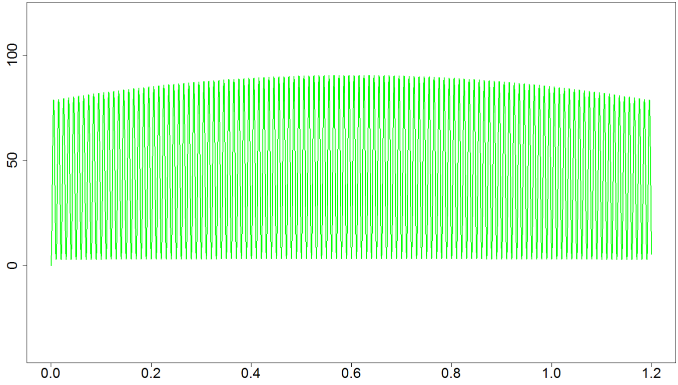

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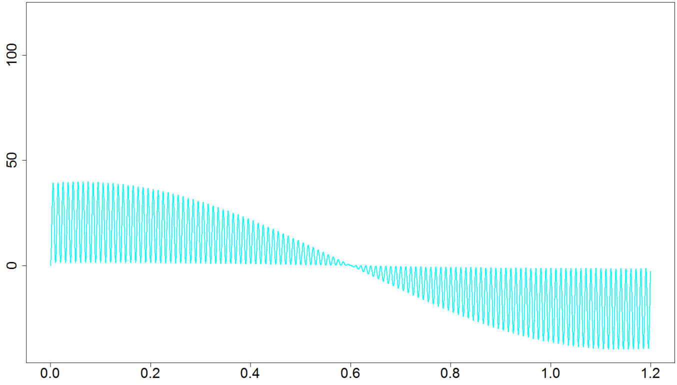

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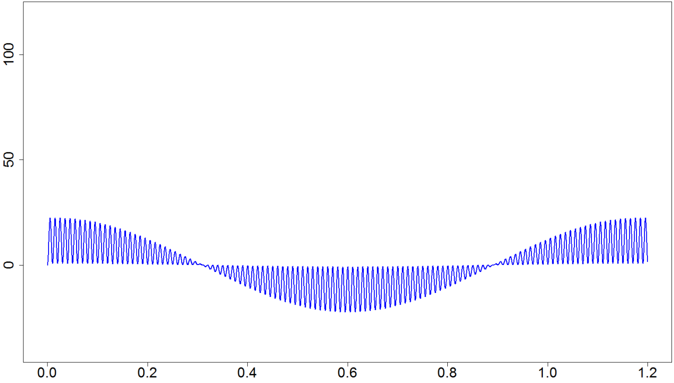

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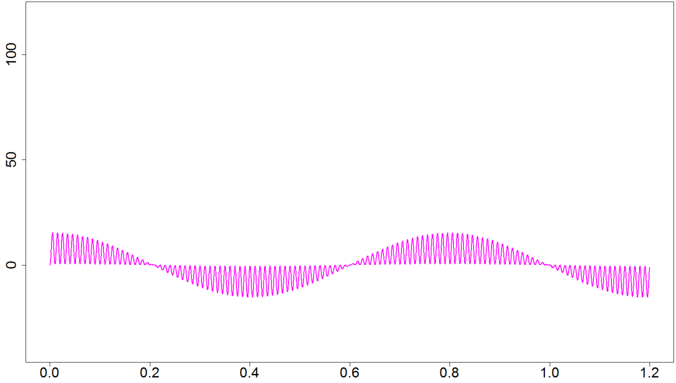

Polynomial chaos expansions of the arc radius R

- Consider here the case $m = 0$, then

$$R^N := \left(Y^N\right)^{\frac{1}{4}} = \sqrt{\sqrt{Y^N}}$$

- The arc radius can be represented as polynomial Chaos expansion:

$$R^N = \sum_{k=0}^{\infty} r_k(t)\Psi_k$$

- Step 1: Series development of $\left(\sqrt{Y^N}\right)_t = \sum_{k=0}^{\infty} \tilde{y}_k(t)\Psi_k$

Step 1: Series development of $\left(\sqrt{Y^N}\right)_t = \sum_{k=0}^{\infty} \tilde{y}_k(t)\Psi_k$:
Approximately it holds

$$\sum_{k=0}^{M} y_k(t)\Psi_k = Y_t^N = \left(\sqrt{Y_t^N}\right)^2 \approx \left(\sum_{k=0}^{M} \tilde{y}_k(t)\Psi_k\right)^2$$

For all $\ell \in \{0, ..., M\}$ it follows

$$y_\ell(t)\,\mathrm{E}\left[\Psi_\ell^2\right] \approx \sum_{j=0}^{M}\sum_{k=0}^{M} \tilde{y}_j(t)\tilde{y}_k(t)\,\mathrm{E}[\Psi_j\Psi_k\Psi_\ell]$$

with $M_{jk\ell} := \frac{\mathrm{E}[\Psi_j\Psi_k\Psi_\ell]}{\mathrm{E}\left[\Psi_\ell^2\right]}$, this results in the nonlinear equation

$$\begin{pmatrix} y_0(t) \\ \vdots \\ y_M(t) \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{M}\sum_{k=0}^{M} M_{jk0}\tilde{y}_j(t)\tilde{y}_k(t) \\ \vdots \\ \sum_{j=0}^{M}\sum_{k=0}^{M} M_{jkM}\tilde{y}_j(t)\tilde{y}_k(t) \end{pmatrix}$$

<u>Step 2:</u> Series development of $R_t^N = \sqrt{\sqrt{Y_t^N}} \approx \sqrt{\sum_{k=0}^M \tilde{y}_k(t)\Psi_k}$ with the same method as in step 1 $\Longrightarrow$

$$R_t^{N,M} = \sum_{k=0}^{M} \tilde{r}_k(t)\Psi_k \qquad\qquad t \in [0,T]$$

as presumed approximation of $\left(R_t^N\right)_{t\in[0,T]}$.

Numerical computations

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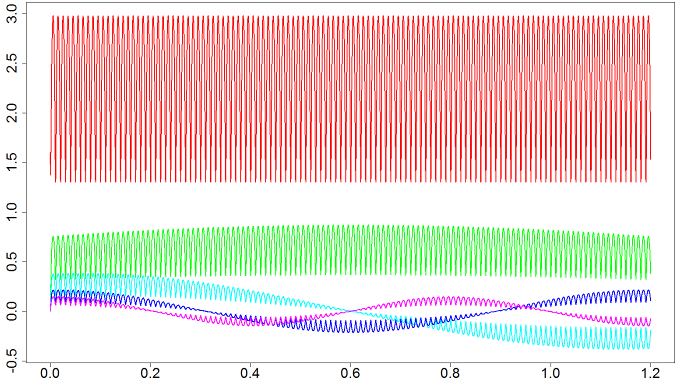

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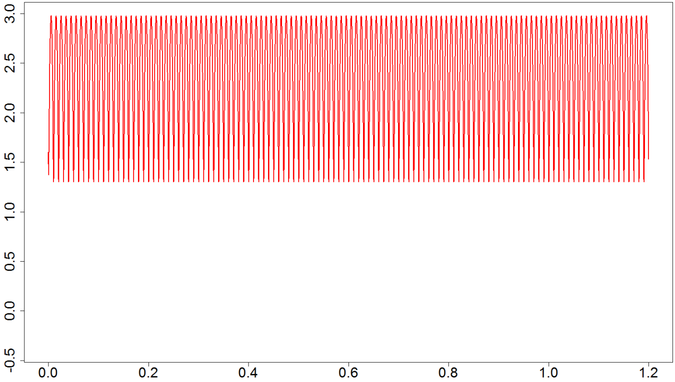

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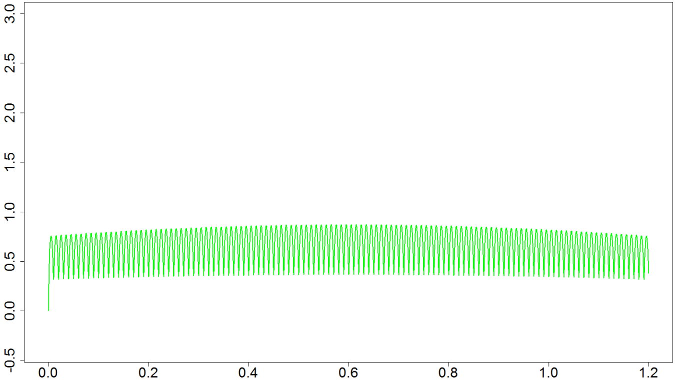

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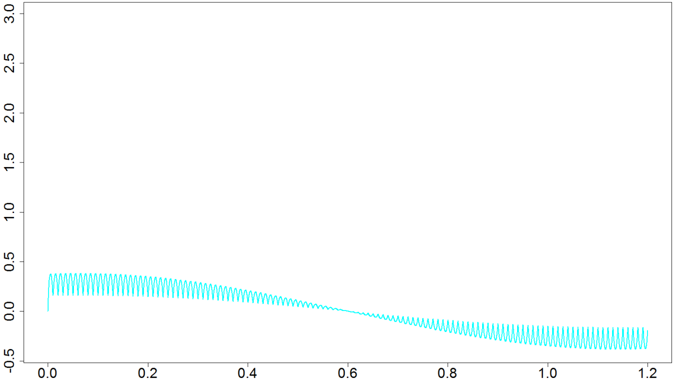

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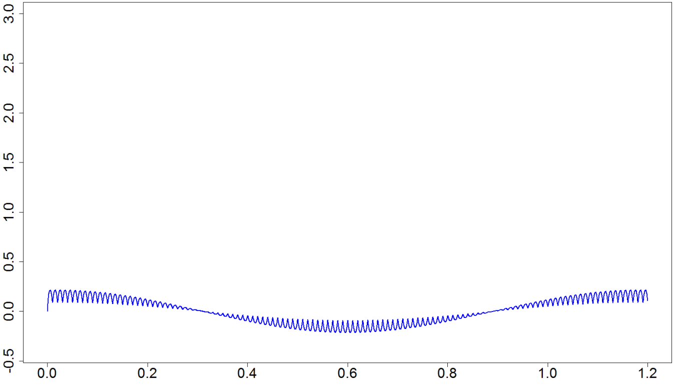

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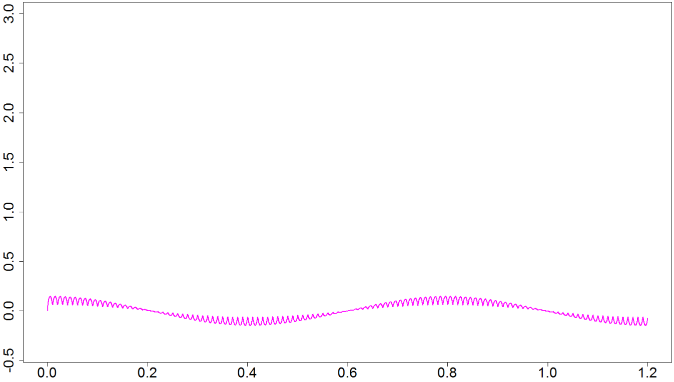
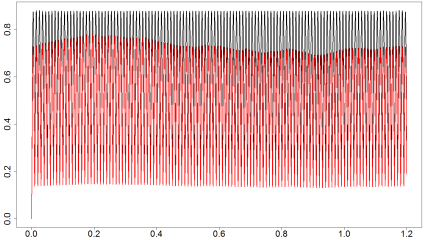


Variance function of the arc radius

- **Black line:** variance function of $R_t^{N,M} = \sum_{k=0}^{M} \tilde{r}_k(t)\Psi_k$:

$$\mathrm{Var}\left[R_t^{N,M}\right] = \sum_{k=0}^{M} \tilde{r}_k^2(t)\,\mathrm{E}\left[\Psi_k^2\right]$$

- **Red line:** estimated variance function of $R^N = \left(Y^N\right)^{\frac{1}{4}}$ by Monte-Carlo method (2000 simulations).



Polynomial chaos expansions of the arc radius U

- Consider here the case $m = 0$, then

$$U_t^N := \frac{k_3 \left(1 + X_t^N\right)^2}{\sqrt{Y_t^N}} i(t)$$

- The arc radius can be represented as polynomial Chaos expansion:

$$U_t^N = \sum_{k=0}^{\infty} u_k(t) \Psi_k$$

- A similiar method as for the arc radius leads to a series development as presumed approximation of U_t^N:

$$U_t^{N,M} = \sum_{k=0}^{M} \tilde{u}_k(t) \Psi_k$$

Numerical computations

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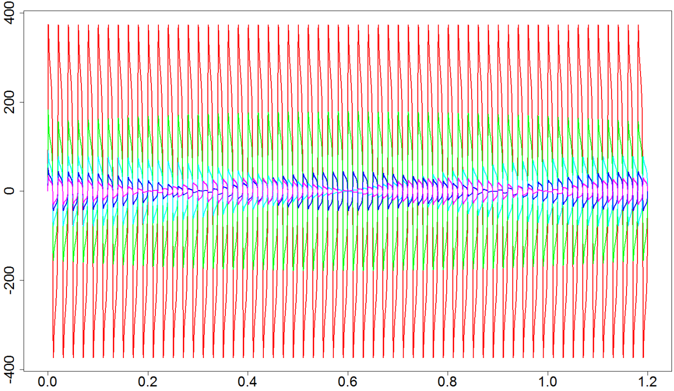

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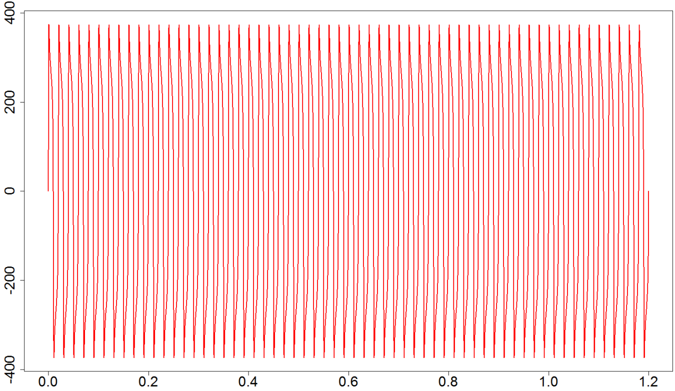

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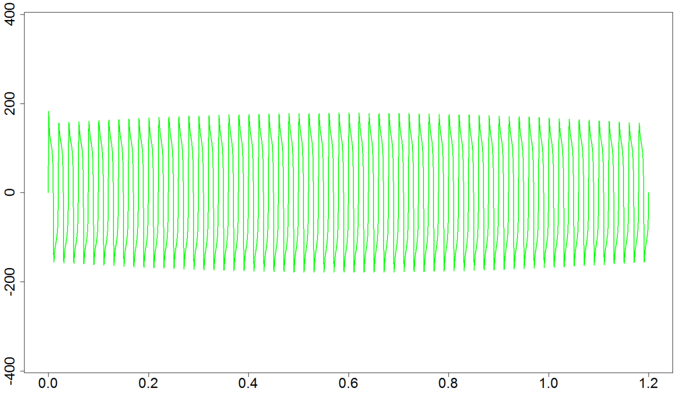

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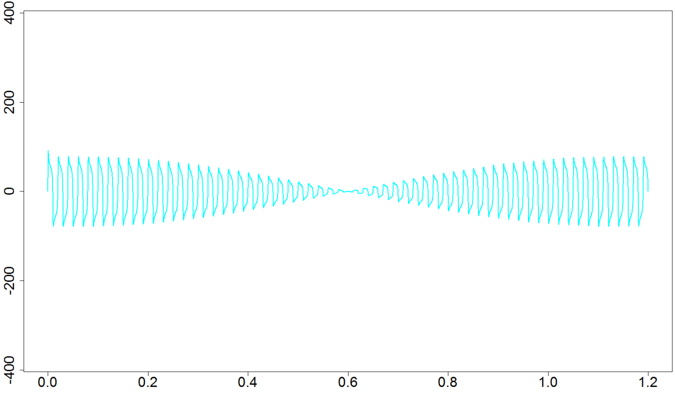

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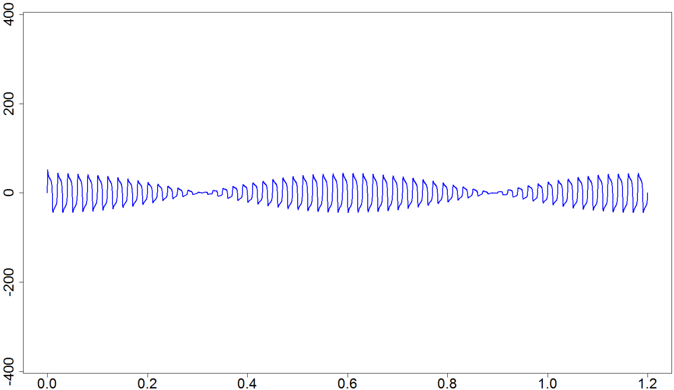

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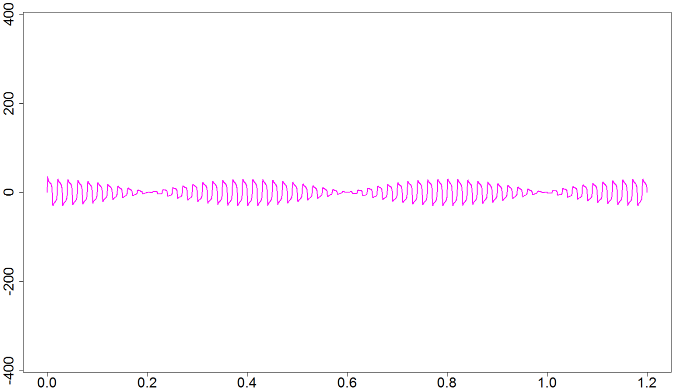
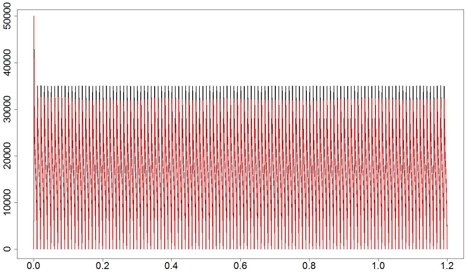


Variance function of the arc voltage

- **Black line:** variance function of $U_t^{N,M} = \sum_{k=0}^{M} \tilde{u}_k(t)\Psi_k$:

$$\mathrm{Var}\left[U_t^{N,M}\right] = \sum_{k=0}^{M} \tilde{u}_k^2(t)\,\mathrm{E}\left[\Psi_k^2\right]$$

- **Red line:** estimated variance function of U^N by Monte-Carlo method (2000 simulations).



Some open questions:

- Characteristics of the stochastic processes $R(t)$ and $U(t)$.
- Statistical analysis of the model.
- Fitting the model to real world data.

List of References

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Thank you for your attention!