Function spaces with varying smoothness

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Zusammenfassung

In dieser Arbeit studieren wir Funktionenräume mit variabler Glattheit. Diese sollen Funktionen klassifizieren, die unterschiedliches Glattheitsverhalten in verschiedenen Gebieten oder einzelnen Punkten besitzen, zum Beispiel Funktionen mit lokalen Singularitäten. Auch spezielle Differentialoperatoren mit Entartungen, beispielsweise auf dem Rand eines Gebietes, benötigen für die mögliche Entwicklung einer Lösungstheorie Funktionenräume, die diese Entartungen reflektieren. Ein Vorläufer solcher Räume vom Sobolev-Typ kann durch die Norm

$$\|u|W_p^{m'}(\mathbb{R}^n)\| + \|\varrho(x)u|W_p^m(\mathbb{R}^n)\|$$

mit $m > m'$ und einer glatten Funktion $\varrho(x)$, die auf einem Gebiet Ω verschwindet, charakterisiert werden. Hier wird von der Funktion u global die Glattheit m' gefordert, jedoch außerhalb von Ω sogar die Glattheit m. In einem allgemeineren Kontext können solche Räume mittels spezieller Pseudodifferentialoperatoren beschrieben werden. Ein solcher Pseudodifferentialoperator hat die Form

$$A(x, D_x)u(x) = (2\pi)^{-n} \int e^{ix\xi} a(x, \xi)\widehat{u}(\xi) d\xi,$$

wobei $\widehat{u}$ die Fouriertransformierte von u und $a(x, \xi)$ das sogenannte Symbol von A bezeichnet. Als Beispiel kann man $a(x, \xi) = \langle\xi\rangle^{\sigma(x)}$ betrachten, wobei $\sigma(x)$ eine reellwertige Funktion aus $S(\mathbb{R}^n)$ ist, die man als variable Glattheit interpretieren kann. Operatoren dieses Typs und die zugehörigen Funktionenräume $W_p^{k,a}(\mathbb{R}^n)$ mit der Norm

$$\|u|L_p\| + \|A^k(x, D_x)u|L_p\|$$

wurden zum Beispiel von Unterberger und Bokobza in [30],[31], Visik und Eskin in [32],[33], Volevic und Kagan in [34] oder Beauzamy in [2] zwischen 1965 und 1972 sowie eine verallgemeinerte Klasse von Pseudodifferentialoperatoren von Beals 1981 in [1] betrachtet. Fast alle in diesen Arbeiten auftauchenden Funktionenräume sind vom Sobolev- oder Besselpotential-Typ. Besov-Räume mit variabler Glattheit wurden zuerst von Leopold 1987 in [13] definiert. Seine Definition der Räume $B_{p,q}^{s,a}(\mathbb{R}^n)$ mit der Norm

$$\|u|B_{p,q}^{s,a}(\mathbb{R}^n)\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \|\varphi_j^a(x, D_x)u|L_p\|^q \right)^{1/q}$$

basiert auf einer Zerlegung $\{\varphi_j^a(x, \xi)\}_{j=0}^{\infty}$ von $\mathbb{R}_x^n \times \mathbb{R}_\xi^n$, die von Symbolen $a(x, \xi)$ geeigneter Pseudodifferentialoperatoren einer bestimmten Klasse erzeugt wird. In den folgenden Jahren veröffentlichte Leopold mehrere Arbeiten zu diesen Räumen, vergleiche [14], [15] und [16], in denen er beispielsweise den Zusammenhang

$$\left(L_p(\mathbb{R}^n), W_p^{k,a}(\mathbb{R}^n)\right)_{\Theta,q} = B_{p,q}^{\Theta k,a}(\mathbb{R}^n)$$

bewies. In [13] ist auch eine Charakterisierung von $B^{s,a}_{p,q}(\mathbb{R}^n)$ mittels Differenzen mit variabler Schrittweite enthalten. Dies war der Ausgangspunkt für Besov, um Funktionenräume mit variabler Glattheit mit Hilfe verschiedener gewichteter Differenzen zu beschreiben, vergleiche [3], [4] und [5]. Es zeigte sich, dass dieser Zugang dieselben Räume $B^{s,a}_{p,q}(\mathbb{R}^n)$ lieferte. Auch eine andere Klasse von Funktionenräumen weist Verbindungen zu diesen Räumen auf. Die Einbettung

$$W_p^{\sigma(x)}(\mathbb{R}^n) \subset L_{q(x)}(\mathbb{R}^n), \text{ wenn } 1 < p \leq \inf_x q(x) \text{ und } \inf_x (s(x) + n/q(x)) > n/p,$$

wobei $W_p^{\sigma(x)}(\mathbb{R}^n)$ ein Spezialfall der Räume $W_p^{1,a}(\mathbb{R}^n)$ ist, vergleiche [16], liefert einen interessanten Zusammenhang zwischen den Räumen mit variabler Glattheit und den Räumen $L_{q(x)}$ mit variabler Integrabilität. Diese Räume wurden zum Beispiel von Kovacik und Rakosnik 1991 in [12] oder später von Samko studiert, vergleiche [20] für Details und mehr Referenzen.
Aktuelles Interesse an Funktionenräumen mit variabler Glattheit gibt es auch aus einer anderen Richtung. Lokale Informationen über das Glattheitsverhalten von Funktionen lassen sich mittels Wavelet-Zerlegungen gewinnen. Eine beliebige Funktion f aus einem Besov-Raum kann als

$$f(x) = \sum_{l,j,m} \lambda^l_{j,m}(f)\Psi^l(2^j x - m)$$

geschrieben werden, wobei Ψ^l fixierte Funktionen mit kompaktem Träger und $\lambda^l_{j,m}(f)$ von f abhängige komplexe Zahlen sind. Auf diesem Weg werden die sogenannten mikrolokalen Räume $C^{s,s'}(x^0)$ dadurch charakterisiert, dass man

$$|\lambda^l_{j,m}(f)| \leq c2^{-js}(1 + |m - 2^j x^0|)^{-s'}$$

für alle $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ und $1 \leq l \leq L \in \mathbb{N}$ fordert. Diese Charakterisierung wurde von Jaffard und Meyer in [11] gegeben, wo diese Räume untersucht wurden. Die Räume $C^{s,s'}(x^0)$ beschreiben das Glattheitsverhalten in einem Punkt $x^0 \in \mathbb{R}^n$ und seiner Umgebung und sind speziell auf die Untersuchung isolierter Singularitäten zugeschnitten, vergleiche [11].
In dieser Arbeit werden wir einen anderen Zugang verfolgen und gehen dabei folgendermaßen vor.
In Abschnitt 2 wiederholen wir grundlegende Definitionen, legen die Notation fest und stellen bekannte Resultate bereit, die wir im Weiteren verwenden.
Die Funktionenräume mit variabler Glattheit $B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$, wobei die Glattheit durch eine Funktion $\mathbb{S} : x \mapsto s(x)$ bestimmt wird und $s_0 \in \mathbb{R}$ die globale Mindestglattheit bezeichnet, definieren wir in Abschnitt 3, zeigen, dass es sich um einen Banachraum handelt und geben einige Grundeigenschaften an. Dann beweisen wir eine äquivalente Norm und mittels dieser können wir klassische Aussagen über punktweise Multiplikatoren und Einbettungen in Besov-Räumen für die Räume $B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$ verallgemeinern.

In den Abschnitten 4 und 5 beschäftigen wir uns mit verschiedenen Wavelet-Zerlegungen. Dabei gehen wir jeweils von bestimmten Zerlegungen aus, die von Triebel in [28] und [29] behandelt wurden, und treffen Aussagen über lokales Verhalten von Funktionen mittels dieser Wavelet-Techniken. Dabei beweisen wir die entscheidenden Hilfsmittel für Abschnitt 6.
In diesem Abschnitt formulieren wir unsere Hauptresultate, die zeigen, dass sich die Räume $B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$ durch spezielle Folgenraumnormen von Waveletkoeffizienten charakterisieren lassen. Das bedeutet, die Kenntnis der Waveletkoeffizienten einer Funktion f gibt Aufschluss über das lokale Glattheitsverhalten von f. Dieser Zusammenhang ist der Schlüssel für die weiteren Untersuchungen. In Abschnitt 6.3 beweisen wir auf diesem Weg, dass die schon erwähnten mikrolokalen Räume $C^{s,s'}(x^0)$ in einem gewissen Sinn mit $B_\infty^{\mathbb{S},s_0}(\mathbb{R}^n)$ zusammenfallen, falls

$$s(x) = \begin{cases} s & : \quad x = x^0 \\ s + s' & : \quad \text{sonst} \end{cases}$$

und $s_0 < 1/p$ gilt.
Im letzten Abschnitt benutzen wir die Charakterisierungen aus Abschnitt 6, um spezielle Probleme zu behandeln. Zum einen zeigen wir, dass die Einbettungen aus Abschnitt 3 scharf sind, und zum anderen geben wir eine Teilantwort auf die folgende interessante Frage: Ist es möglich für ein vorgegebenes Glattheitsverhalten $s(x)$ eine Funktion f zu konstruieren, die genau dieses Verhalten aufweist? Für ein spezielles $s(x)$ geben wir eine explizite Konstruktion für eine solche Funktion f an.

1 Introduction

We study function spaces with varying smoothness. These spaces are supposed to classify functions with different smoothness behavior in different domains or points, for example functions with local singularities. Also special differential operators with degenerations, for instance at the boundary of a domain, require function spaces that reflect these degenerations. A forerunner of such spaces, of Sobolev-type, can be characterized by the norm

$$\|u|W_p^{m'}(\mathbb{R}^n)\| + \|\varrho(x)u|W_p^m(\mathbb{R}^n)\|$$

with $m > m'$ and a smooth funktion $\varrho(x)$ that vanishes on a domain Ω. Here the function u has to satisfy the smoothness degree m' globaly, but outside of Ω even the degree m. From a more general point of view, such spaces can be described by using special pseudodifferential operators. Such operators are defined by

$$A(x, D_x)u(x) = (2\pi)^{-n} \int e^{ix\xi} a(x,\xi)\widehat{u}(\xi)d\xi,$$

where $\widehat{u}$ denotes the Fourier transform of u and $a(x,\xi)$ is the so-called symbol of A. As an example, one can study the case $a(x,\xi) = \langle\xi\rangle^{\sigma(x)}$, where $\sigma(x)$ is a real valued function belonging to $S(\mathbb{R}^n)$ that can be interpreted as varying smoothness. Operators of this type and the corresponding function spaces $W_p^{k,a}(\mathbb{R}^n)$ with the norm

$$\|u|L_p\| + \|A^k(x, D_x)u|L_p\|$$

have been studied, for example, by Unterberger and Bokobza in [30],[31], Visik and Eskin in [32],[33], Volevic and Kagan in [34] or Beauzamy in [2] between 1965 and 1972 as well as a more general class of pseudodifferential operators by Beals 1981 in [1]. Almost all function spaces that appeared in these papers were of Sobolev- or Besselpotential-type. Besov spaces with varying smoothness were first defined by Leopold 1987 in [13]. His definition of the spaces $B_{p,q}^{s,a}(\mathbb{R}^n)$ with the norm

$$\|u|B_{p,q}^{s,a}(\mathbb{R}^n)\| = \left(\sum_{j=0}^{\infty} 2^{jsq}\|\varphi_j^a(x, D_x)u|L_p\|^q\right)^{1/q}$$

is based on a resolution $\{\varphi_j^a(x,\xi)\}_{j=0}^{\infty}$ of $\mathbb{R}_x^n \times \mathbb{R}_\xi^n$, that is induced by symbols $a(x,\xi)$ of suitable pseudodifferential operators belonging to a certain class. Thereafter, Leopold published several papers concerning these spaces, see [14], [15] and [16], in which, for instance, he proved the relation

$$\left(L_p(\mathbb{R}^n), W_p^{k,a}(\mathbb{R}^n)\right)_{\Theta,q} = B_{p,q}^{\Theta k,a}(\mathbb{R}^n).$$

His dissertation [13] also contains a characterization of $B_{p,q}^{s,a}(\mathbb{R}^n)$ in terms of differences with variable steps. That was the starting point from which Besov described function spaces of varying smoothness by means of differently weighted differences, see [3], [4] and [5]. It turned out that this approach produced the same spaces $B^{s,a}_{p,q}(\mathbb{R}^n)$. There is another class of function spaces having connections to these spaces. The embedding

$$W^{\sigma(x)}_p(\mathbb{R}^n) \subset L_{q(x)}(\mathbb{R}^n), \text{ if } 1 < p \leq \inf_x q(x) \text{ and } \inf_x (s(x) + n/q(x)) > n/p,$$

where $W^{\sigma(x)}_p(\mathbb{R}^n)$ is a special case of spaces $W^{1,a}_p(\mathbb{R}^n)$, see [16], gives an interesting relation between the spaces with varying smoothness and the spaces $L_{q(x)}$ with varying integrability. These spaces have been studied, for example, by Kovacik and Rakosnik 1991 in [12] or later on by Samko, see [20] for details and more references.
There is also current interest on function spaces with varying smoothness from another point of view. It is possible to get local information about the smoothness behavior of a function by using wavelet techniques. An arbitrary function f belonging to a Besov space can be written as

$$f(x) = \sum_{l,j,m} \lambda^l_{j,m}(f)\Psi^l(2^j x - m),$$

where Ψ^l are fixed funktions with compact support and $\lambda^l_{j,m}(f)$ are complex numbers depending on f. In this way the so-called two-microlocal spaces $C^{s,s'}(x^0)$ can be characterized by demanding

$$|\lambda^l_{j,m}(f)| \leq c2^{-js}(1 + |m - 2^j x^0|)^{-s'}$$

for all $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ and $1 \leq l \leq L \in \mathbb{N}$. This characterization was given by Jaffard and Meyer in [11], where these spaces were studied. The spaces $C^{s,s'}(x^0)$ describe the smoothness behavior at a point $x^0 \in \mathbb{R}^n$ and its neighborhood. They are preferrently used for consideration of local singularities, see [11].
We choose a different approach for our investigations. The plan of this work is the following.
We start by recalling basic definitions in section 2. Thereafter, we fix the notation and collect some known results that we will use in the sequel.
In section 3, we define function spaces of varying smoothness $B^{\mathbb{S},s_0}_p(\mathbb{R}^n)$, where the function $\mathbb{S} : x \mapsto s(x)$ determines the smoothness pointwise and $s_0 \in \mathbb{R}$ is the global smoothness parameter. Then we prove that this space is a Banach space and give some basic properties. After that we provide an equivalent norm in $B^{\mathbb{S},s_0}_p(\mathbb{R}^n)$, which enables us to generalize classical assertions about pointwise multipliers and embeddings in Besov spaces for the spaces of varying smoothness.
In the sections 4 and 5 we study different wavelet decompositions. The starting points are decompositions that have been treated by Triebel in [28] and [29]. We prove some assertions concerning local behavior of functions using these wavelet techniques and provide the main tools for section 6.

In this section we formulate our main results. That is to say, we characterize the spaces $B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$ by using special sequence space norms of wavelet coefficients. That means, that the knowledge about the wavelet coefficients of a function f gives information about the local smoothness behavior of f. This relation is the key for our further investigaions. Using it, we prove in section 6.3 that the two-microlocal spaces $C^{s,s'}(x^0)$ mentioned above are in some sense equal to $B_\infty^{\mathbb{S},s_0}(\mathbb{R}^n)$, if

$$s(x) = \begin{cases} s & : \quad x = x^0 \\ s + s' & : \quad \text{otherwise} \end{cases}$$

and $s_0 < 1/p$ hold.
In the last section we use the characterizations from section 6 to treat specific problems. As the first problem, we show that the embeddings from section 3 are optimal. The second problem concerns the following interesting question: Given smoothness behavior $s(x)$, is it possible to construct a function f that satisties this behavior exactly? We give a partial answer by explicitely constructing such a function for a special chosen $s(x)$.

2 Preliminaries

In this section we provide all definitions, results and the notation that we shall use in the sequel. For the proofs, the references will be given. In the first part we consider function spaces on $\mathbb{R}^n$ and recall some results that we need as important tools throughout the work. The same is done in the second part of this section for spaces on domains.

2.1 Function Spaces on $\mathbb{R}^n$

We start by briefly recalling the definition of Besov spaces on $\mathbb{R}^n$. We follow the Fourier-analytical approach.
Let $\varphi_0 \in S(\mathbb{R}^n)$ with

$$\varphi_0(x) = 1 \text{ if } |x| \leq 1 \quad \text{and} \quad \varphi_0(x) = 0 \text{ if } |x| \geq 3/2. \tag{2.1}$$

We put

$$\varphi(x) = \varphi_0(x) - \varphi_0(2x) \quad \text{and} \quad \varphi_j(x) = \varphi(2^{-j}x) \tag{2.2}$$

for $j \in \mathbb{N}$. Then

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1$$

and $(\varphi_j)_{j=0}^{\infty}$ is called a dyadic resolution of unity in $\mathbb{R}^n$. We denote by $\widehat{\varphi}$ the Fourier transform of φ, and by $\varphi^{\vee}$, its inverse Fourier transform.

Definition 2.1 *Let $0 < p \leq \infty$, $s \in \mathbb{R}$ and $(\varphi_j)_{j=0}^{\infty}$ be the above resolution of unity. Then*

$$B^s_p(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \|f|B^s_p(\mathbb{R}^n)\| = \left(\sum_{j=0}^{\infty} 2^{jsp} \|(\varphi_j \widehat{f})^{\vee}|L_p(\mathbb{R}^n)\|^p \right)^{1/p} < \infty \right\}$$

with the usual modification for $p = \infty$.

This is the well-known definition of ordinary Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ for the special case $p = q$. These spaces have been introduced by O.V. Besov in 1959/60 for $s > 0$, $1 < p < \infty$ and $1 \leq q < \infty$ in terms of derivatives and differences. The Fourier analytical characterization is due to J. Peetre 1967, and was extended to the full range for s and p in 1973 also by J. Peetre. There are many books and papers dealing with these spaces, we refer to [22] for a detailed description of the properties of $B^s_{p,q}(\mathbb{R}^n)$ and a list of references. Later we shall give some properties of the Besov spaces explicitely, but only those we need for our purpose. Here we remark that the so-defined spaces are quasi-Banach spaces (Banach spaces for $1 \leq p \leq \infty$) that are independent of the given resolution of unity $(\varphi_j)_{j=0}^{\infty}$.

Another space for which we give the definition here is the so-called Hölder-Zygmund space $\mathcal{C}^s(\mathbb{R}^n)$, first introduced by Zygmund 1945 as a generalization of the Hölder spaces. For a function $f \in L_p(\mathbb{R}^n)$ we define the well-known differences by

$$\Delta_h^1 f(x) = f(x+h) - f(x) \quad \text{and} \quad \Delta_h^{k+1} f(x) = \Delta_h^1(\Delta_h^k f(x))$$

for $k \in \mathbb{N}$ and $x, h \in \mathbb{R}^n$.

Definition 2.2 *Let $s > 0$ and $k \in \mathbb{N}$ with $k > s$, then*

$$\mathcal{C}^s(\mathbb{R}^n) = \{ f \in C(\mathbb{R}^n) : \|f|\mathcal{C}^s(\mathbb{R}^n)\| < \infty \},$$

where

$$\|f|\mathcal{C}^s(\mathbb{R}^n)\| = \|f|C(\mathbb{R}^n)\| + \sup_{\substack{x \in \mathbb{R}^n \\ 0 < |h| \le 1}} |h|^{-s} |\Delta_h^k f(x)|.$$

Here the space $C(\mathbb{R}^n)$ is the usually normed space of bounded, uniformly continuous functions. The space $\mathcal{C}^s(\mathbb{R}^n)$ fits in the scale of the Besov spaces in the following way

$$\mathcal{C}^s(\mathbb{R}^n) = B^s_\infty(\mathbb{R}^n) \quad \text{for } s > 0.$$

Now, we list the properties of the Besov spaces for which we shall prove counterparts for the spaces of varying smoothness in the corresponding sections. We start to recall a result concerning pointwise multipliers where we need both the Besov spaces and the Hölder-Zygmund spaces defined above. For the proof we refer to Theorem 2.8.2. and the following Corollary in [22].

Theorem 2.1 *Let $0 < p \le \infty$, $s \in \mathbb{R}$ and*

$$\varrho > \max\left(s, n\Big(\frac{1}{\min(p,1)} - 1\Big) - s\right).$$

Then every $g \in \mathcal{C}^\varrho(\mathbb{R}^n)$ is a multiplier for $B^s_p(\mathbb{R}^n)$. In other words, $f \to gf$ yields a bounded linear mapping from $B^s_p(\mathbb{R}^n)$ into itself, and there exists a positive constant c such that

$$\|gf|B^s_p(\mathbb{R}^n)\| \le c\|g|\mathcal{C}^\varrho(\mathbb{R}^n)\| \, \|f|B^s_p(\mathbb{R}^n)\|$$

holds for all $g \in \mathcal{C}^\varrho(\mathbb{R}^n)$ and all $f \in B^s_p(\mathbb{R}^n)$.

Of course, pointwise multiplication in general must be interpreted in the distributional sense, see 2.8.1. in [22]. The next assertion concerns embeddings between Besov spaces with different metrics.

Theorem 2.2 *Let* $0 < p_1 \leq p_2 \leq \infty$ *and* $-\infty < s_2 \leq s_1 < \infty$. *Then*

$$B^{s_1}_{p_1}(\mathbb{R}^n) \subset B^{s_2}_{p_2}(\mathbb{R}^n) \quad \text{if and only if} \quad s_1 - \frac{n}{p_1} \geq s_2 - \frac{n}{p_2}.$$

The proof is covered by Theorem 2.2.3 in [19]. The following Theorem gives different equivalent norms for the Besov spaces and is related to the mapping property of the operator $I - \Delta$, where I denotes the Identity and

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

is the Laplacian. For a real number σ the operator $(I - \Delta)^\sigma$ is defined by

$$(I - \Delta)^\sigma f = ((1 + |x|^2)^\sigma \widehat{f})^\vee \quad \text{for} \quad f \in S'(\mathbb{R}^n).$$

Theorem 2.3 *Let* $-\infty < s < \infty$, $m \in \mathbb{N}$, $0 < p \leq \infty$ *and* $\tau < s$. *If* $\sigma \in \mathbb{R}$ *then* $(I - \Delta)^\sigma$ *maps* $B^s_p(\mathbb{R}^n)$ *isomorphically onto* $B^{s-2\sigma}_p(\mathbb{R}^n)$ *and the following expressions*

$$\|(I - \Delta)^\sigma f | B^{s-2\sigma}_p(\mathbb{R}^n)\|, \tag{2.3}$$

$$\sum_{|\alpha| \leq m} \|D^\alpha f | B^{s-m}_p(\mathbb{R}^n)\|, \tag{2.4}$$

$$\text{and} \quad \|f | B^\tau_p(\mathbb{R}^n)\| + \sum_{|\alpha| = m} \|D^\alpha f | B^{s-m}_p(\mathbb{R}^n)\| \tag{2.5}$$

are equivalent quasi-norms in $B^s_p(\mathbb{R}^n)$.

The formulas (2.3) and (2.4) are given by Theorem 2.3.8.(i) in [22]. As in the proof given there, one can use Fourier Multipliers to prove the equivalent norm (2.5).
The last two assertions we want to recall in this subsection concern dilation properties of the Besov spaces.

Proposition 2.1 *Let* $0 < p \leq \infty$ *and* $s < 0$. *Then there exists a constant* $c > 0$ *such that for all* $\lambda \in (0, 1]$,

$$\|f(\lambda \cdot) | B^s_p(\mathbb{R}^n)\| \leq c \lambda^{s - n/p} \|f | B^s_p(\mathbb{R}^n)\| \quad \text{for all } f \in B^s_p(\mathbb{R}^n).$$

Proposition 2.2 *Let* $0 < p \leq \infty$ *and* $\infty > s > \max(0, n(1/p - 1))$. *Then there exists a constant* $c > 0$ *such that for all* $\lambda \in (0, 1]$,

$$\|f(\lambda^{-1} \cdot) | B^s_p(\mathbb{R}^n)\| \leq c \lambda^{-(s - n/p)} \|f | B^s_p(\mathbb{R}^n)\| \quad \text{for all } f \in B^s_p(\mathbb{R}^n).$$

These two propositions can be found in [8], 2.3.1, but the second one already appeared in [22], 3.4.1.

2.2 Function Spaces on Domains

To define the Besov spaces on domains we follow the usual procedure of restriction. Although it is possible to work with much more general domains, we consider only bounded domains in $\mathbb{R}^n$ with smooth boundaries because later on we shall work with balls only.

Definition 2.3 *Let Ω be a domain in $\mathbb{R}^n$, $0 < p \leq \infty$ and $s \in \mathbb{R}$, then*

$$B^s_p(\Omega) = \{f \in S'(\mathbb{R}^n) : \text{ there exists } g \in B^s_p(\mathbb{R}^n) \text{ with } g|\Omega = f\} \tag{2.6}$$

with

$$\|f|B^s_p(\Omega)\| = \inf \|g|B^s_p(\mathbb{R}^n)\|,$$

where the Infimum is taken over all g in the sense of (2.6).

Now, we list the counterparts of the corresponding properties of $B^s_p(\mathbb{R}^n)$ for the spaces on Ω.

Theorem 2.4 *Let $0 < p \leq \infty$, $s \in \mathbb{R}$ and*

$$\varrho > \max\left(s, n\Big(\frac{1}{\min(p,1)} - 1\Big) - s\right).$$

Then every $g \in \mathcal{C}^\varrho(\Omega)$ is a multiplier for $B^s_p(\Omega)$. In other words, $f \to gf$ yields a bounded linear mapping from $B^s_p(\Omega)$ into itself and there exists a positive constant c such that

$$\|gf|B^s_p(\Omega)\| \leq c\|g|\mathcal{C}^\varrho(\Omega)\| \; \|f|B^s_p(\Omega)\|$$

holds for all $g \in \mathcal{C}^\varrho(\Omega)$ and all $f \in B^s_p(\Omega)$.

We refer to 3.3.2. in [22].

Theorem 2.5 *Let $0 < p_1, p_2 \leq \infty$ and $-\infty < s_2 \leq s_1 < \infty$. Then*

$$B^{s_1}_{p_1}(\Omega) \subset B^{s_2}_{p_2}(\Omega) \quad \text{if and only if} \quad s_1 - \frac{n}{p_1} \geq s_2 - \frac{n}{p_2}.$$

The proof is covered by 2.4.4 in [19].

Theorem 2.6 *Let $-\infty < s < \infty$, $m \in \mathbb{N}$, $0 < p \leq \infty$ and $\tau \leq s$. Then the following expressions*

$$\sum_{|\alpha| \leq m} \|D^\alpha f|B^{s-m}_p(\Omega)\|, \tag{2.7}$$

$$\|f|B^\tau_p(\Omega)\| + \sum_{|\alpha| = m} \|D^\alpha f|B^{s-m}_p(\Omega)\| \tag{2.8}$$

are equivalent quasi-norms in $B^s_p(\Omega)$.

Formula (2.7) is given by Theorem 3.3.5. in [22]. Formula (2.8) can be proved indirectly. We sketch the method here. First one proves, that

$$\|f|B_p^{s-m}(\Omega)\| + \sum_{|\alpha|=m} \|D^\alpha f|B_p^{s-m}(\Omega)\| \tag{2.9}$$

is an equivalent quasi-norm in $B_p^s(\Omega)$ by using (2.7). One side of the desired estimate is obvious. To prove the other direction we assume the converse, that is to say there exists a sequence $(f_j)_{j=1}^\infty$ such that

$$1 = \|D^\alpha f_j|B_p^{s-m}(\Omega)\| \geq j\Big(\|f_j|B_p^{s-m}(\Omega)\| + \sum_{|\alpha|=m} \|D^\alpha f_j|B_p^{s-m}(\Omega)\|\Big)$$

holds for $0 < |\alpha| < m$. Because $B_p^s(\Omega)$ is compactly embedded into $B_p^{s-1}(\Omega)$ we find that the sequence $(f_j)_{j=1}^\infty$ converges in both spaces to a function f. The norm of f in $B_p^{s-1}(\Omega)$ is greater or equal to one, because of our assumption. On the other hand the norm of f_j in $B_p^{s-m}(\Omega)$ tends to zero as j tends to infinity. That is a contradiction which proves the desired direction. Finally, to obtain formula (2.8) from (2.9) one has to follow a very similar idea.
In all the assertions stated so far, constants appeared that may depend on Ω in different ways. As we will see in section 3, we need to control these dependencies of the constants on Ω. As already mentioned we shall work with balls, typically they are centered in a point $x \in \mathbb{R}^n$ and have the radius 2^{-K} for a natural number K. By an easy translation argument the point x does not influence the constants at all. Therefore we are interested in the influence of K. A very important property of Besov spaces on domains in this sense is the so-called homogeneity property. We denote by $B_{x,r}$ the ball with radius $r > 0$ centered in $x \in \mathbb{R}^n$. In the case $x = 0$ we omit it and write only B_r.

Proposition 2.3 *Let $1 < p \leq \infty$, $-\infty < s < 1/p$ and $0 < \lambda \leq 1$, then*

$$\|f(\lambda\cdot)|B_p^s(B_1)\| \sim \lambda^{s-n/p}\|f|B_p^s(B_\lambda)\|, \tag{2.10}$$

where the equivalence constants are independent of f and λ.

For the proof we refer to 3.9(iii) in [25]. Because we will use it very intensively throughout the work, we add a short discussion about this remarkable property.
Discussion: Let $0 > s_2 > s_1$ and $1 < p \leq \infty$. Putting $\lambda = 2^{-K}$ we have by formula (2.10)

$$\begin{aligned} 2^{-K(s_1-n/p)}\|f|B_p^{s_1}(B_{2^{-K}})\| &\sim \|f(2^{-K}\cdot)|B_p^{s_1}(B_1)\| \\ &\leq c\|f(2^{-K}\cdot)|B_p^{s_2}(B_1)\| \\ &\sim 2^{-K(s_2-n/p)}\|f|B_p^{s_2}(B_{2^{-K}})\| \end{aligned}$$

and, hence,

$$\|f|B_p^{s_1}(B_{2^{-K}})\| \leq c2^{-K(s_2-s_1)}\|f|B_p^{s_2}(B_{2^{-K}})\|, \tag{2.11}$$

or

$$2^{-K(s_1-s_0)}\|f|B_p^{s_1}(B_{2^{-K}})\| \leq c2^{-K(s_2-s_0)}\|f|B_p^{s_2}(B_{2^{-K}})\|, \tag{2.12}$$

for a real number s_0 where the constant c is independent of f and K. These estimates reflect a rather typical situation for our work, and show how to control the dependence of the constants on K. That is to say, first we shift the problem to the ball $B_{x,1}$, then we use known results for Besov spaces on domains and finally shift back to $B_{x,2^{-K}}$. Unfortunately this strategy is obviously restricted to $s < 1/p$, but one direction of (2.10) can be generalized without any restriction for $s \in \mathbb{R}$.

Proposition 2.4 *Let $1 < p \leq \infty$, $s \in \mathbb{R}$ and $0 < \lambda \leq 1$, then*

$$\|f(\lambda\cdot)|B_p^s(B_1)\| \geq c\lambda^{s-n/p}\|f|B_p^s(B_\lambda)\|, \tag{2.13}$$

where the equivalence constants are independent of f and λ.

For $s > 0$ this is an easy consequence of Definition 2.3 and Proposition 2.2. The only case not yet covered is $s = 0$ and $p = \infty$. But, in that case, we can prove it directly with the same idea as was used in the proof of Proposition 1, 3.4.1., in [22]. In any case, the first step of the above strategy can be maintained for all s. In 3.2 we will describe how to maintain also the last step.
In section 3 it will become clear that the dependence of the constants on the smoothness parameter s, which depends on x and K, must be controlled for all the previous estimates too. In some cases this can be done directly by observing the constants appearing in the original proofs, for which the references were given. But in most cases we can use interpolation arguments to ensure that if $s_1 \leq s \leq s_2$ for two real numbers s_1, s_2 then the constants can be chosen independent of s.

Remark 2.1 *From now on we denote by the symbol c all kinds of real numbers with different dependencies, but they are always meant to be independent of f, x and K.*

3 The space $B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$

We start this section by roughly describing our motivation. Suppose we have been given two functions f_1, f_2 with the following properties. Let f_1 have a singularity at a point $x^0 \in \mathbb{R}^n$ such that $f_1 \in B_p^s(\mathbb{R}^n)$ for some p and s but $f_1 \notin B_p^{s+\varepsilon}(\mathbb{R}^n)$ for any $\varepsilon > 0$. In all points $x \neq x^0$ let f_1 be a smooth function. Now let f_2 have many singularities of the above type. Then both functions belong to the same Besov space even though f_1 is much smoother than f_2 in the sense that $f_1 \in C^\infty(\mathbb{R}^n \setminus U_{x^0})$ for every neigborhood U_{x^0} of x^0. Our aim is to construct a scale of function spaces with which we are able to distinguish between f_1 and f_2 for example, which means that such a space should reflect pointwise smoothness behavior of its elements. In this section we give the definition of this scale of spaces, prove some basic properties and look at an example.

3.1 Definition and basic assertions

To take pointwise smoothness behavior into account, we need a function that gives for every $x \in \mathbb{R}^n$ a smoothness value $s(x)$. Such a function should somehow represent the typical situation, where pointwise jumps to lower smoothness levels are allowed. The definition for the appropriate class reads as follows.

Definition 3.1 *A real-valued function $\mathbb{S} : x \mapsto s(x)$ on $\mathbb{R}^n$ is called lower semi-continuous, if for any $t \in \mathbb{R}$*

$$\Omega_t = \{x \in \mathbb{R}^n : s(x) > t\}$$

is an open set.

It is easy to verify that such a function has the following property.
A real valued-function $\mathbb{S} : x \mapsto s(x)$ on $\mathbb{R}^n$ is lower semi-continuous if, and only if, for any $x^0 \in \mathbb{R}^n$ and any $\varepsilon > 0$ there is a number $\tau = \tau(x^0, \varepsilon) > 0$ such that

$$\inf_{|x-x^0|\le\tau} s(x) \le s(x^0) \le \varepsilon + \inf_{|x-x^0|\le\tau} s(x), \tag{3.1}$$

see also in [10] (p.242).

Remark 3.1 *In the following we will use bounded lower semi-continuous functions $\mathbb{S}$, that means*

$$-\infty < s_{\min} = \inf_{y\in\mathbb{R}^n} s(y) \le s(x) \le \sup_{y\in\mathbb{R}^n} s(y) = s_{\max} < \infty. \tag{3.2}$$

We put

$$s_{K,x} = \inf_{|y-x|\le 2^{-K+2}} s(y)$$

for $x \in \mathbb{R}^n$ and $K \in \mathbb{N}$, which may increase in K for a fixed x. The reason why the radius that influences $s_{K,x}$ is chosen as 2^{-K+2} we shall explain in 4.1.3. Now we define the main object of our work, the space $B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$.

Definition 3.2 *Let $1 < p \leq \infty$ and let $\mathbb{S}$ be a bounded lower semi-continuous function in $\mathbb{R}^n$ with $s_{\min} \geq s_0$ for a real number s_0. Then*

$$B_p^{\mathbb{S},s_0}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \|f|B_p^{\mathbb{S},s_0}(\mathbb{R}^n)\| < \infty \right\},$$

where

$$\|f|B_p^{\mathbb{S},s_0}(\mathbb{R}^n)\| = \|f|B_p^{s_0}(\mathbb{R}^n)\| + \sup_{x\in\mathbb{R}^n} \sup_{K\in\mathbb{N}} 2^{-K(s_{K,x}-s_0)} \|f|B_p^{s_{K,x}}(B_{x,2^{-K}})\|. \quad (3.3)$$

First we note that the supremum in the above norm is taken over x and K where the smoothness parameter and the balls depend on these numbers. That is the reason for the discussion and the considerations about the constants in the Preliminaries. Let us describe what happens in the norm. The first term checks the global smoothness of a given function f, where the supremum term concerns local improvements by the following procedure. For a fixed point $x \in \mathbb{R}^n$ we consider a ball centered in x with radius 2^{-K} and ask if f belongs to the Besov space with smoothness $s_{K,x} \geq s_0$ in this ball. Now we increase K and therefore shrink the ball around x and ask the same question again with respect to a possibly higher degree of smoothness. We continue this procedure for all K, then all x, and finally check if the supremum over all these norms multiplied by the weight factor $2^{-K(s_{K,x}-s_0)}$ is finite. The question arises: why we use this specific weight? Looking at (2.11) we see that this factor appears in a natural way when we compare different smoothness levels.
In the sequel we formulate some basic properties of these spaces.

Theorem 3.1 *Let $1 < p \leq \infty$ and let $\mathbb{S}$ be a bounded lower semi-continuous function in $\mathbb{R}^n$. Then $B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$ is a Banach space.*

Proof Obviously $B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$ is a normed space. We prove the completeness. Let $\{f_l\}_{l=1}^{\infty}$ be a Cauchy sequence in $B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$. Then it is also a Cauchy sequence in $B_p^{s_0}(\mathbb{R}^n)$. Because $B_p^{s_0}(\mathbb{R}^n)$ is a complete space, it contains a function f with

$$\|f - f_l|B_p^{s_0}(\mathbb{R}^n)\| \longrightarrow 0 \quad \text{for } l \to \infty. \quad (3.4)$$

It is sufficient to prove that

$$\|f - f_l|B_p^{\mathbb{S},s_0}(\mathbb{R}^n)\| \longrightarrow 0 \quad \text{for } l \to \infty, \quad (3.5)$$

because then $f \in B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$ and therefore $B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$ would be complete. Because of (3.4) in order to prove (3.5) it is even enough to show that

$$\sup_{x\in\mathbb{R}^n} \sup_{K\in\mathbb{N}} 2^{-K(s_{K,x}-s_0)} \|f - f_l|B_p^{s_{K,x}}(B_{x,2^{-K}})\| \longrightarrow 0 \quad \text{for } l \to \infty. \quad (3.6)$$

Let $\varepsilon > 0$ be given, then for fixed x, K we have by triangle inequality

$$2^{-K(s_{K,x}-s_0)}\|f - f_l|B_p^{s_{K,x}}(B_{x,2^{-K}})\| \\ \leq \quad \|f_m - f_l|B_p^{\mathbb{S},s_0}(\mathbb{R}^n)\| + 2^{-K(s_{K,x}-s_0)}\|f - f_m|B_p^{s_{K,x}}(B_{x,2^{-K}})\|, \qquad (3.7)$$

where the first term is smaller than $\varepsilon/2$ for $l, m \geq l_0(\varepsilon/2)$ because $\{f_l\}_{l=1}^{\infty}$ is a Cauchy sequence in $B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$. But that also means $\{f_l\}_{l=1}^{\infty}$ is a Cauchy sequence in $B_p^{s_{K,x}}(B_{x,2^{-K}})$. This space is complete and, therefore, contains a function $g_{x,K}$ with

$$\|g_{x,K} - f_m|B_p^{s_{K,x}}(B_{x,2^{-K}})\| \leq \frac{\varepsilon}{2} \qquad \text{for } \; m \geq m_0(\varepsilon/2, x, K).$$

Because the limit element is unique, the restriction of f to $B_{x,2^{-K}}$ is equal to $g_{x,K}$. Looking at (3.7) we see that

$$2^{-K(s_{K,x}-s_0)}\|f - f_l|B_p^{s_{K,x}}(B_{x,2^{-K}})\| \leq \varepsilon \qquad \text{for } \; l \geq l_0(\varepsilon/2),$$

which also proves (3.6), because l_0 was chosen independent of x and K.

□

The next property follows directly from the Definition of $B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$.

Proposition 3.1 *Let $1 < p \leq \infty$ and let $\mathbb{S}$ be a bounded lower semi-continuous function in $\mathbb{R}^n$ with $s_{\min} \geq s_0 \geq s_1$. Then*

$$B_p^{\mathbb{S},s_0}(\mathbb{R}^n) \subset B_p^{s_0}(\mathbb{R}^n) \quad \textit{and} \quad B_p^{\mathbb{S},s_0}(\mathbb{R}^n) \subset B_p^{\mathbb{S},s_1}(\mathbb{R}^n).$$

Now we discuss an example that we treat later again.

Example 3.1 *Let δ be the Dirac-distribution, defined by*

$$\delta(\varphi) = \varphi(0) \qquad \textit{for } \varphi \in S(\mathbb{R}^n)$$

and let $\mathbb{S} : x \mapsto s(x)$ be a bounded lower semi-continuous function in $\mathbb{R}^n$ with $s(0) < n/p - n$. Then

$$\delta \in B_p^{\mathbb{S},s_0}(\mathbb{R}^n).$$

Proof We use standard arguments for the first term of (3.3) and get

$$\|\delta|B_p^{s_0}(\mathbb{R}^n)\| = \left(\sum_{j=0}^{\infty} 2^{js_0p}\|(\varphi_j\widehat{\delta})^\vee|L_p(\mathbb{R}^n)\|^p\right)^{1/p}$$

$$
\begin{aligned}
&\sim \left(\sum_{j=0}^{\infty} 2^{js_0p} 2^{j(np-n)} \|\varphi^{\vee}|L_p(\mathbb{R}^n)\|^p \right)^{1/p} \\
&\sim \left(\sum_{j=0}^{\infty} 2^{jp(s_0+n-n/p)} \right)^{1/p} \qquad (3.8)\\
&\leq c \left(\sum_{j=0}^{\infty} 2^{jp(s(0)+n-n/p)} \right)^{1/p} < \infty .
\end{aligned}
$$

For the second term of (3.3) we treat as a first case all K, x with $0 \notin B_{x,2^{-K+1}}$. But then we have

$$\|\delta|B_p^{s_{K,x}}(B_{x,2^{-K}})\| = 0.$$

In the second case we treat all K, x with $0 \in B_{x,2^{-K+1}}$. Then $s_{K,x} \leq s(0)$ and because $2^{-K(s_{K,x}-s_0)} \leq 1$ we can estimate

$$
\begin{aligned}
2^{-K(s_{K,x}-s_0)}\|\delta|B_p^{s_{K,x}}(B_{x,2^{-K}})\| &\leq \|\delta|B_p^{s_{K,x}}(B_{0,2^{-K+2}})\| \\
&\leq \|\delta|B_p^{s(0)}(B_{0,2^{-K+2}})\| \\
&\leq \|\delta|B_p^{s(0)}(\mathbb{R}^n)\| \leq c.
\end{aligned}
$$

That shows that the second term of (3.3) is finite.

□

That result follows our expectation exactly, that is to say that for the Dirac-distribution we need only a restriction on the smoothness function at the origin. We will calculate more examples later with the help of decomposition techniques. Here we add the following assertion.

Proposition 3.2 *Let $1 < p \leq \infty$ and $\mathbb{S} : x \mapsto s(x)$ be a bounded lower semi-continuous function in $\mathbb{R}^n$. Then for $K_0 \in \mathbb{N}$*

$$\|f|B_p^{s_0}(\mathbb{R}^n)\| + \sup_{x\in\mathbb{R}^n} \sup_{K \geq K_0} 2^{-K(s_{K,x}-s_0)}\|f|B_p^{s_{K,x}}(B_{x,2^{-K}})\|$$

is an equivalent norm in $B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$.

This shows that only large values of K, corresponding to small balls, are of interest.

Proof One direction is obvious. To prove the converse it is enough to show that

$$\sup_{x\in\mathbb{R}^n} 2^{-K(s_{K,x}-s_0)}\|f|B_p^{s_{K,x}}(B_{x,2^{-K}})\| \leq c \sup_{x\in\mathbb{R}^n} 2^{-K_0(s_{K_0,x}-s_0)}\|f|B_p^{s_{K_0,x}}(B_{x,2^{-K_0}})\|$$

holds for $K < K_0$. Therefore we choose points $x_l \in B_{x,2^{-K}}$ for $l = 1, \ldots, L \in \mathbb{N}$ with the property

$$B_{x,2^{-K}} \subset \bigcup_{l=1}^{L} B_{x_l,2^{-K_0}}.$$

Then by means of this covering we can prove

$$2^{-K(s_{K,x}-s_0)}\|f|B_p^{s_{K,x}}(B_{x,2^{-K}})\| \le c2^{-K(s_{K,x}-s_0)}\sum_{l=1}^{L}\|f|B_p^{s_{K,x}}(B_{x_l,2^{-K_0}})$$

by a procedure of extension and restriction because we have $s_{K,x} \le s_{K_0,x_l}$ for all $l = 1,\dots,L \in \mathbb{N}$. Now we estimate further

$$\begin{aligned}2^{-K(s_{K,x}-s_0)}\|f|B_p^{s_{K,x}}(B_{x,2^{-K}})\| &\le c2^{-K(s_{K,x}-s_0)}L\|f|B_p^{s_{K_0,y}}(B_{y,2^{-K_0}})\| \\ &\le c2^{-K_0(s_{K_0,y}-s_0)}\|f|B_p^{s_{K_0,y}}(B_{y,2^{-K_0}})\|,\end{aligned}$$

where we used $2^{-K(s_{K,x}-s_0)} \le 1$ and $2^{K_0(s_{K_0,y}-s_0)} \le 2^{K_0(s_{\max}-s_0)} \le c$ and chose $y \in \{x_1,\dots,x_L\}$ such that $\|f|B_p^{s_{K_0,x_l}}(B_{x_l,2^{-K_0}})\|$ is maximal. Taking now the supremum over all $\mathbb{R}^n$ on both sides we arrive at the desired estimate.

□

3.2 An equivalent norm

Now we provide a tool that enables us to preserve the second step of the strategy discussed in the Preliminaries also but without restrictions on the smoothness function $s(x)$.

Theorem 3.2 *Let $1 < p \le \infty$ and $\mathbb{S} : x \mapsto s(x)$ be a bounded lower semi-continuous function in $\mathbb{R}^n$ with $s_{\max} - m < 1/p$ for a natural number m. Then for $s_0 < 1/p$*

$$\|f|B_p^{s_0}(\mathbb{R}^n)\| + \sup_{K\in\mathbb{N},x\in\mathbb{R}^n} 2^{-K(s_{K,x}-s_0)}\sum_{|\alpha|=m}\|D^\alpha f|B_p^{s_{K,x}-m}(B_{x,2^{-K}})\|. \tag{3.9}$$

is an equivalent norm in $B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$.

Proof *Step 1*
We start to estimate the supremum in (3.9) from above. By Definition we have

$$\sum_{|\alpha|=m}\|D^\alpha f|B_p^{s_{K,x}-m}(B_{x,2^{-K}})\| = \sum_{|\alpha|=m}\inf_{g_\alpha}\|g_\alpha|B_p^{s_{K,x}-m}(\mathbb{R}^n)\|,$$

where the infimum is taken over all g_α with $g_\alpha|_{B_{x,2^{-K}}} = D^\alpha f$. If we allow only functions h in the infimum for which even $h|_{B_{x,2^{-K}}} = f$ holds we have

$$\begin{aligned}\sum_{|\alpha|=m}\|D^\alpha f|B_p^{s_{K,x}-m}(B_{x,2^{-K}})\| &\le \sum_{|\alpha|=m}\inf_h\|D^\alpha h|B_p^{s_{K,x}-m}(\mathbb{R}^n)\| \\ &\le \inf_h\sum_{|\alpha|=m}\|D^\alpha h|B_p^{s_{K,x}-m}(\mathbb{R}^n)\|.\end{aligned}$$

But now by formula (2.5) we arrive at

$$\sum_{|\alpha|=m} \|D^\alpha f|B_p^{s_{K,x}-m}(B_{x,2^{-K}})\| \leq \inf_h c\|h|B_p^{s_{K,x}}(\mathbb{R}^n)\| = c\|f|B_p^{s_{K,x}}(B_{x,2^{-K}})\|.$$

<u>*Step 2*</u>
For the opposite direction we have by formula (2.13)

$$2^{-K(s_{K,x}-s_0)}\|f|B_p^{s_{K,x}}(B_{x,2^{-K}})\| \leq c2^{K(s_0-n/p)}\|f(2^{-K}\cdot)|B_p^{s_{K,x}}(B_{x,1})\|. \qquad (3.10)$$

We treat the norm on the right-hand side. By applying formula (2.8) we get

$$\begin{aligned}\|f(2^{-K}\cdot)|B_p^{s_{K,x}}(B_{x,1})\| \;\leq\; & c\|f(2^{-K}\cdot)|B_p^{\tau}(B_{x,1})\| \\ & +c2^{-mK}\sum_{|\alpha|=m}\|(D^\alpha f)(2^{-K}\cdot)|B_p^{s_{K,x}-m}(B_{x,1})\|,\end{aligned}$$

where we used

$$D^\alpha[f(2^{-K}\cdot)] = 2^{-|\alpha|K}(D^\alpha f)(2^{-K}\cdot). \qquad (3.11)$$

If we choose $\tau = s_0$ we can use the homogeneity property (2.10) for both terms because $s_0 < 1/p$ and $s_{K,x} - m < 1/p$. We obtain

$$\begin{aligned}\|f(2^{-K}\cdot)|B_p^{s_{K,x}}(B_{x,1})\| \;\leq\; & c2^{-K(s_0-n/p)}\|f|B_p^{s_0}(B_{x,2^{-K}})\| \\ & +c2^{-K(s_{K,x}-m-n/p+m)}\sum_{|\alpha|=m}\|D^\alpha f|B_p^{s_{K,x}-m}(B_{x,2^{-K}})\|\end{aligned}$$

and arrive by a simple embedding argument at

$$\begin{aligned}\|f(2^{-K}\cdot)|B_p^{s_{K,x}}(B_{x,1})\| \;\leq\; & c2^{-K(s_0-n/p)}\|f|B_p^{s_0}(\mathbb{R}^n)\| \\ & +c2^{-K(s_{K,x}-n/p)}\sum_{|\alpha|=m}\|D^\alpha f|B_p^{s_{K,x}-m}(B_{x,2^{-K}})\|,\end{aligned} \qquad (3.12)$$

which inserted into (3.10) gives

$$\begin{aligned}2^{-K(s_{K,x}-s_0)}\|f|B_p^{s_{K,x}}(B_{x,2^{-K}})\| \;\leq\; & c\|f|B_p^{s_0}(\mathbb{R}^n)\| \\ & +c2^{-K(s_{K,x}-s_0)}\sum_{|\alpha|=m}\|D^\alpha f|B_p^{s_{K,x}-m}(B_{x,2^{-K}})\|,\end{aligned}$$

which is the desired estimate to prove the second direction of (3.9).

□

The strategy sketched before Proposition 2.4 now works as follows. By this equivalent norm we can lift the smoothness level for the supremum terms below $1/p$ on the ball $B_{x,1}$ and with the help of (2.10) shift the problem we want to prove back to the ball $B_{x,2^{-K}}$.

3.3 Further properties

In this section we generalize the results we stated in the Preliminaries for the usual Besov spaces to the spaces with varying smoothness. We follow exactly the strategy mentioned in the previous sections.

3.3.1 Pointwise multipliers

The aim of this subsection is to generalize Theorem 2.1.

Theorem 3.3 *Let* $1 < p \leq \infty$ *and let* $\mathbb{S}$ *be a bounded lower semi-continuous function in* $\mathbb{R}^n$*. Let* $s_0 < 1/p$*, then* $g \in \mathcal{C}^\varrho(\mathbb{R}^n)$ *with* $\varrho > \max(s_{\max}, -s_0)$ *is a pointwise multiplier for* $B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$*. In other words,* $f \mapsto gf$ *yields a bounded linear mapping from* $B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$ *into itself and there exists a constant* $c > 0$ *such that*

$$\|gf|B_p^{\mathbb{S},s_0}(\mathbb{R}^n)\| \leq c\|g|\mathcal{C}^\varrho(\mathbb{R}^n)\| \; \|f|B_p^{\mathbb{S},s_0}(\mathbb{R}^n)\|$$

holds for all $g \in \mathcal{C}^\varrho(\mathbb{R}^n)$ *and all* $f \in B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$*.*

Proof We only have to take care about the Supremum in (3.3) because for the first part we have

$$\|gf|B_p^{s_0}(\mathbb{R}^n)\| \leq c\|g|\mathcal{C}^\varrho(\mathbb{R}^n)\| \; \|f|B_p^{s_0}(\mathbb{R}^n)\| \tag{3.13}$$

for $\varrho > -s_0$ by Theorem 2.1. For the term inside of the Supremum by using the formulas (2.13) and Theorem 2.4 with $\varrho > s_{\max}$ we get

$$\begin{aligned} 2^{-K(s_{K,x}-s_0)}&\|gf|B_p^{s_{K,x}}(B_{x,2^{-K}})\| \\ &\leq c2^{K(s_0-n/p)}\|(gf)(2^{-K}\cdot)|B_p^{s_{K,x}}(B_{x,1})\| \\ &\leq c2^{K(s_0-n/p)}\|g(2^{-K}\cdot)|\mathcal{C}^\varrho(B_{x,1})\| \; \|f(2^{-K}\cdot)|B_p^{s_{K,x}}(B_{x,1})\|. \end{aligned}$$

Looking at Definition 2.2, it is easy to varify, that

$$\|g(2^{-K}\cdot)|\mathcal{C}^\varrho(B_{x,1})\| \leq \|g|\mathcal{C}^\varrho(\mathbb{R}^n)\|$$

because in these spaces we only deal with differences. Therefore we have

$$\begin{aligned} 2^{-K(s_{K,x}-s_0)}&\|gf|B_p^{s_{K,x}}(B_{x,2^{-K}})\| \\ &\leq c2^{K(s_0-n/p)}\|g|\mathcal{C}^\varrho(\mathbb{R}^n)\| \; \|f(2^{-K}\cdot)|B_p^{s_{K,x}}(B_{x,1})\|. \end{aligned} \tag{3.14}$$

In the proof of Theorem 3.2 we find formula (3.12), insert it into (3.14) and obtain

$$\begin{aligned} 2^{-K(s_{K,x}-s_0)}&\|gf|B_p^{s_{K,x}}(B_{x,2^{-K}})\| \\ &\leq c\|g|\mathcal{C}^\varrho(\mathbb{R}^n)\|\Big(\|f|B_p^{s_0}(\mathbb{R}^n)\| + c2^{-K(s_{K,x}-s_0)}\sum_{|\alpha|=m}\|D^\alpha f|B_p^{s_{K,x}-m}(B_{x,2^{-K}})\|\Big). \end{aligned}$$

If we now take the supremum over x and K on both sides we finally arrive by Theorem 3.2 at

$$\sup_{x,K} 2^{-K(s_{K,x}-s_0)} \|gf|B_p^{s_{K,x}}(B_{x,2^{-K}})\| \leq c\|g|\mathcal{C}^{\varrho}(\mathbb{R}^n)\| \; \|f|B_p^{\mathbb{S},s_0}(\mathbb{R}^n)\|,$$

which proves the desired assertion.

□

3.3.2 Embeddings

The main goal in this subsection is to generalize Theorem 2.2.

Theorem 3.4 *Let $1 < p_1 \leq p_2 \leq \infty$ and let $\mathbb{S}^1$ and $\mathbb{S}^2$ be bounded lower semi-continuous functions in $\mathbb{R}^n$. Then for $s_0^1, s_0^2 < 1/p$*

$$B_{p_1}^{\mathbb{S}^1,s_0^1}(\mathbb{R}^n) \subset B_{p_2}^{\mathbb{S}^2,s_0^2}(\mathbb{R}^n) \qquad \textit{if} \qquad s^1(x) - \frac{n}{p_1} \geq s^2(x) - \frac{n}{p_2} \quad \textit{for all } x \in \mathbb{R}^n$$
$$\textit{and} \qquad s_0^1 - \frac{n}{p_1} \geq s_0^2 - \frac{n}{p_2}.$$

Proof Theorem 2.2 gives the desired estimate for the first term of the norm (3.3). For the second term we use formula (2.13) and obtain by applying Theorem 2.5

$$\begin{aligned} 2^{-K(s^2_{K,x}-s^2_0)}\|f|B_{p_2}^{s^2_{K,x}}(B_{x,2^{-K}})\| &\leq c2^{K(s^2_0-n/p_2)}\|f(2^{-K}\cdot)|B_{p_2}^{s^2_{K,x}}(B_{x,1})\| \\ &\leq c2^{K(s^2_0-n/p_2)}\|f(2^{-K}\cdot)|B_{p_1}^{s^1_{K,x}}(B_{x,1})\|. \end{aligned}$$

Now we use formula (3.12) from the proof of Theorem 3.2 again to get

$$\begin{aligned} &2^{-K(s^2_{K,x}-s^2_0)}\|f|B_{p_2}^{s^2_{K,x}}(B_{x,2^{-K}})\| \\ &\leq c2^{K(s^2_0-n/p_2-s^1_0+n/p_1)}\|f|B_{p_1}^{s^1_0}(\mathbb{R}^n)\| \\ &\qquad +c2^{-K(s^1_{K,x}-n/p_1-s^2_0+n/p_2)}\sum_{|\alpha|=m}\|D^\alpha f|B_p^{s^1_{K,x}-m}(B_{x,2^{-K}})\| \\ &\leq c\|f|B_{p_1}^{s^1_0}(\mathbb{R}^n)\| + c2^{-K(s^1_{K,x}-s^1_0)}\sum_{|\alpha|=m}\|D^\alpha f|B_p^{s^1_{K,x}-m}(B_{x,2^{-K}})\|. \end{aligned}$$

After taking the supremum over x, K on both sides we arrive by Theorem 3.2 at

$$\sup_{x,K} 2^{-K(s^2_{K,x}-s^2_0)}\|f|B_{p_2}^{s^2_{K,x}}(B_{x,2^{-K}})\| \leq c\|f|B_{p_1}^{\mathbb{S}^1,s^1_0}(\mathbb{R}^n)\|,$$

which proves the assertion.

□

Corollary 3.1 *Let* $1 < p \leq \infty$ *and let* $\mathbb{S}^1$ *and* $\mathbb{S}^2$ *be bounded lower semi-continuous functions in* $\mathbb{R}^n$*. Then for* $s_0^1, s_0^2 < 1/p$

$$B_p^{\mathbb{S}^1, s_0^1}(\mathbb{R}^n) \subset B_p^{\mathbb{S}^2, s_0^2}(\mathbb{R}^n) \qquad \textit{if} \qquad s^1(x) \geq s^2(x) \textit{ for all } x \in \mathbb{R}^n$$
$$\textit{and} \qquad s_0^1 \geq s_0^2.$$

This follows from the last Theorem for the special case $p_1 = p_2$.

Corollary 3.2 *Let* $1 < p \leq \infty$ *and let* $\mathbb{S}$*,* $\mathbb{S}^1$ *and* $\mathbb{S}^2$ *be bounded lower semi-continuous functions in* $\mathbb{R}^n$*. Then for* $f_1 \in B_p^{\mathbb{S}^1, s_0^1}(\mathbb{R}^n)$ *and* $f_2 \in B_p^{\mathbb{S}^2, s_0^2}(\mathbb{R}^n)$ *follows, that for* $s_0^1, s_0^2 < 1/p$

$$f_1 + f_2 \in B_p^{\mathbb{S}, s_0}(\mathbb{R}^n) \qquad \textit{with} \qquad s(x) \leq \min(s^1(x), s^2(x))$$
$$\textit{and} \qquad s_0 \leq \min(s_0^1, s_0^2).$$

This follows immediately by triangle inequality and Corollary 3.1.

4 Decomposition with C^∞-wavelets

Already some years ago procedures were established to reduce problems in function spaces to the level of sequence spaces with the help of decomposition techniques. There are many different possible ways to do so, for example by using molecules, atoms, quarks and wavelets. We want to use special wavelet decompositions developed by Triebel in [28] and [29] to characterize the space $B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$ in terms of the wavelet coefficients of its elements. This will be done in section 6. In this section we discuss a decomposition by C^∞-wavelets for $s < 0$, its generalization for all s and investigate how to use these decompositions to get local smoothness information. All notation in this section is based on [28].

4.1 Wavelet-frames for distributions

Here we only consider the case $s < 0$.

4.1.1 Definition and Theorem

We define

$$\mathbb{R}^n_{++} = \{y \in \mathbb{R}^n : y = (y_1, \dots, y_n), y_i > 0\}.$$

Let k be a non-negative C^∞-function in $\mathbb{R}^n$ with

$$\operatorname{supp} k \subset \{y \in \mathbb{R}^n : |y| < 2^J\} \cap \mathbb{R}^n_{++} \tag{4.1}$$

for some $J \in \mathbb{N}$ and

$$\sum_{m \in \mathbb{Z}^n} k(x - m) = 1. \tag{4.2}$$

For $\beta \in \mathbb{N}_0^n$, we put $k^\beta(x) = (2^{-J}x)^\beta k(x) \geq 0$ and define the local means of $f \in S'(\mathbb{R}^n)$ with respect to $k^\beta(x)$ by

$$k^\beta(t, f)(x) = \int_{\mathbb{R}^n} k^\beta(y) f(x + ty) dy, \quad t > 0, \quad x \in \mathbb{R}^n \tag{4.3}$$

and

$$k^\beta_{j,m}(f) = k^\beta(2^{-j}, f)(2^{-j}m), \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n, \tag{4.4}$$

interpreted in the distributional sense. We abbreviate

$$\sum_{\beta \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} = \sum_{\beta,j,m},$$

and define for $s \in \mathbb{R}$ the following norm

$$\|k(f)|l_p\|_s = \left(\sum_{\beta,j,m} 2^{j(s-n/p)p} |k^\beta_{j,m}(f)|^p \right)^{1/p}. \tag{4.5}$$

In addition let $\omega \in S(\mathbb{R}^n)$ with $\operatorname{supp} \omega \subset (-\pi, \pi)^n$ and $\omega(x) = 1$ if $|x| \leq 2$. Then we define

$$\omega^\beta(x) = \frac{i^{|\beta|} 2^{J|\beta|}}{(2\pi)^n \beta!} x^\beta \omega(x), \tag{4.6}$$

with $|\beta| = \beta_1 + \cdots + \beta_n$ and $\beta! = \beta_1! \cdots \beta_n!$. Let

$$\Omega^\beta(x) = \sum_{m \in \mathbb{Z}^n} (\omega^\beta)^\vee(m) e^{-imx}. \tag{4.7}$$

Definition 4.1 *Let φ_0, φ be given by (2.1) and (2.2). Then the mother wavelets $\Phi^\beta_M(x)$ and the father wavelets $\Phi^\beta_F(x)$ are given by*

$$(\Phi^\beta_M)^\vee(\xi) = \varphi(\xi)\Omega^\beta(\xi), \quad \xi \in \mathbb{R}^n$$

and

$$(\Phi^\beta_F)^\vee(\xi) = \varphi_0(\xi)\Omega^\beta(\xi), \quad \xi \in \mathbb{R}^n.$$

Remark 4.1 *For all $\alpha \in \mathbb{N}^n_0$, the following holds*

$$\int_{\mathbb{R}^n} \Phi^\beta_M(\xi)\xi^\alpha d\xi = 0.$$

Furthermore, Φ^β_M and Φ^β_F are entire analytic functions and we have

$$\Phi^\beta_M(x) = \sum_{m \in \mathbb{Z}^n} (\omega^\beta)^\vee(m)\widehat{\varphi}(x+m), \quad x \in \mathbb{R}^n \tag{4.8}$$

and

$$\Phi^\beta_F(x) = \sum_{m \in \mathbb{Z}^n} (\omega^\beta)^\vee(m)\widehat{\varphi}_0(x+m), \quad x \in \mathbb{R}^n. \tag{4.9}$$

We put

$$\Phi^\beta_{j,m}(x) = \begin{cases} \Phi^\beta_F(x-m) & : \quad j = 0 \\ \Phi^\beta_M(2^j x - m) & : \quad j \in \mathbb{N} \end{cases} \tag{4.10}$$

and

$$\mu = \{\mu^\beta_{j,m} \in \mathbb{C} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n, \beta \in \mathbb{N}^n_0\},$$

and define $\|\mu|l_p\|_s$ in the same way as in (4.5).

Theorem 4.1 *Let $1 < p \leq \infty$ and $s < 0$.*
(i) *Then $f \in S'(\mathbb{R}^n)$ is an element of $B^s_p(\mathbb{R}^n)$ if, and only if, it can be represented as*

$$f = \sum_{\beta,j,m} \mu^\beta_{j,m} \Phi^\beta_{j,m}$$

with $\|\mu|l_p\|_s < \infty$*, with unconditional convergence in* $S'(\mathbb{R}^n)$*. Furthermore,*

$$\|f|B^s_p(\mathbb{R}^n)\| \sim \inf \|\mu|l_p\|_s,$$

where the infimum is taken over all admissible representations.
(ii) *Any* $f \in \bigcup_{s<0} B^s_\infty(\mathbb{R}^n)$ *can be represented as*

$$f = \sum_{\beta,j,m} k^\beta_{j,m}(f)\Phi^\beta_{j,m}, \tag{4.11}$$

unconditional convergence in $S'(\mathbb{R}^n)$*. In addition,* $f \in B^s_p(\mathbb{R}^n)$ *if, and only if,* $\|k(f)|l_p\|_s < \infty$*.*
(iii) *Let* $f \in B^s_p(\mathbb{R}^n)$*. Then*

$$\|f|B^s_p(\mathbb{R}^n)\| \sim \|k(f)|l_p\|_s,$$

where the equivalence constants are independent of f*, this means that the coefficients* $k^\beta_{j,m}(f)$ *give an optimal representation.*

The proof of this Theorem was given in 2003 by H. Triebel in [28], Theorem 2. This kind of decomposition stands in a certain sense in contrast to the quarkonial decompositions, because the functions $\Phi^\beta_{j,m}$, as building blocks, are not compactly supported, but the coefficients $k^\beta_{j,m}(f)$ are local in the sense that we only need information about the function f in a small ball around $2^{-j}m$. This fact gives us one possibility to describe local smoothness behavior of a function in terms of its coefficients $k^\beta_{j,m}(f)$, which will be done in 4.1.3.

4.1.2 Examples

(a) Let δ be the Dirac-distribution again, then we try to discover the known results about which Besov spaces contain δ. Here we can calculate the coefficients explicitely and get

$$k^\beta_{j,m}(\delta) = 2^{jn}(-2^{-J}m)^\beta k(-m), \qquad \text{for } m \in \operatorname{supp} k,$$

therefore

$$\|k(\delta)|l_p\|_s = \left(\sum_{\beta,j,m} 2^{j(s-n/p)p}2^{jnp}|(2^{-J}m)^\beta|^p k(-m)^p\right)^{1/p}.$$

Because of $m \in \operatorname{supp} k$, we have $|2^{-J}m| \le q < 1$, if we put $q = \sup_{m\in\operatorname{supp} k} |2^{-J}m|$. Then the sum over β is a geometric series. The remaining sum over m is finite and we obtain

$$\|k(\delta)|l_p\|_s \sim \left(\sum_{j=0}^\infty 2^{j(s-n/p)p}2^{jnp}\right)^{1/p}, \tag{4.12}$$

which corresponds to formula (3.8). Hence, we have the well-known necessary and sufficient conditions

$$\delta \in B^s_p(\mathbb{R}^n), \quad \text{if, and only if,} \quad s < \frac{n}{p} - n, \text{ for } p < \infty$$

and

$$\delta \in B^s_\infty(\mathbb{R}^n), \quad \text{if, and only if,} \quad s \leq -n.$$

(b) Let $\delta^{(\gamma)} = D^\gamma \delta$ with δ as in (a) and a multiindex γ. If we now calculate the coefficients we get

$$k^\beta_{j,m}(\delta^{(\gamma)}) = (-1)^{|\gamma|} 2^{jn} 2^{j|\gamma|} \sum_{\substack{\gamma'+\gamma''=\gamma \\ \gamma' \leq \beta}} c(-2^{-J}m)^{\beta-\gamma'} (D^{\gamma''}k)(-m), \quad \text{for } m \in \operatorname{supp} k,$$

where $\gamma' \leq \beta$ means $\gamma'_i \leq \beta_i$ for $i = 1, \ldots, n$. Hence, we have after estimation from above

$$\|k(\delta^{(\gamma)})|l_p\|_s \leq c \left(\sum_{\beta,j,m} 2^{j(s-n/p)p} 2^{jnp} 2^{j|\gamma|p} |2^{-J}m|^{(\beta-\gamma')p} \right)^{1/p}.$$

Now, with arguments similar to those in example (a), we obtain

$$\|k(\delta^{(\gamma)})|l_p\|_s \leq c \left(\sum_{j=0}^{\infty} 2^{j(s-n/p)p} 2^{jnp} 2^{j|\gamma|p} \right)^{1/p}$$

and can state

$$\delta^{(\gamma)} \in B^s_p(\mathbb{R}^n), \quad \text{if} \quad s < \frac{n}{p} - n - |\gamma|.$$

(c) Let $g(x) = \psi(x)|x|^{-\alpha}$ for $\psi \in C^\infty(\mathbb{R}^n)$ with $\operatorname{supp} \psi \subset \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $\alpha \in \mathbb{R}$ with $n - 1 < \alpha < n$. We put

$$f(x) = (D^\gamma g)(x) \quad \text{for } \gamma \in \mathbb{N}^n_0 \text{ with } |\gamma| = 1.$$

Now we estimate the coefficients $k^\beta_{j,m}(f)$ from above. By definition is

$$k^\beta_{j,m}(f) = \int_{\substack{y \in \operatorname{supp} k \\ |2^{-j}m + 2^{-j}y| \leq 1}} (2^{-J}y)^\beta k(y) f(2^{-j}m + 2^{-j}y) dy.$$

If we put $q = \sup_{y \in \operatorname{supp} k} |2^{-J}y|$, then $q < 1$ holds. As a first case we only consider all m with $|m| \geq 2^{J+1}$. It follows $|2^{-j}m + 2^{-j}y| \geq 2^{J-j}$ and $f(2^{-j}m + 2^{-j}y) \in L^{loc}_1$. Additionally we know

$$2^{-j}(|m| - 2^J) \leq |2^{-j}m + 2^{-j}y|.$$

If also $|m| \geq 2^{j+1}$ for $j > J$ holds, then

$$\{y \in \mathbb{R}^n : |y| \leq 2^J\} \cap \{y \in \mathbb{R}^n : |2^{-j}m + 2^{-j}y| \leq 1\} = \emptyset.$$

This means that in the first case we can restrict ourselves to all m with the property $2^{J+1} \leq |m| \leq 2^{j+1}$. Then it follows

$$|k^{\beta}_{j,m}(f)| \leq cq^{|\beta|} \int |D^{\gamma}(\psi(2^{-j}m + 2^{-j}y)|2^{-j}m + 2^{-j}y|^{-\alpha})|dy,$$

where we integrate only over all y with

$$2^{-j}(|m| - 2^J) \leq |2^{-j}m + 2^{-j}y| \leq 2^{-j}(|m| + 2^J).$$

That gives us

$$|k^{\beta}_{j,m}(f)| \leq cq^{|\beta|}2^{jn}[2^{-j}(|m| - 2^J)]^{n-\alpha-1} = cq^{|\beta|}2^{j(\alpha+1)}(|m| - 2^J)^{n-\alpha-1}. \quad (4.13)$$

In the remaining case $|m| \leq 2^{J+1}$ we integrate in the distributional sense. Therefore we write

$$k^{\beta}_{j,m}(f) = 2^{jn} \int\limits_{\substack{|y|\leq 1 \\ (2^j y - m) \in \operatorname{supp} k}} (2^{j-J}y - 2^{-J}m)^{\beta} k(2^j y - m) f(y) dy.$$

This time we put

$$q' = \sup_{\substack{y \in \mathbb{R}^n \\ (2^j y - m) \in \operatorname{supp} k}} |2^{j-J}y - 2^{-J}m|,$$

then also $q' < 1$ holds, and it follows that

$$|k^{\beta}_{j,m}(f)| \leq 2^{jn} q'^{|\beta|} 2^j \int |\psi(y)||y|^{-\alpha}|(D^{\gamma}k)(2^j y - m)|dy.$$

By $|2^j y - m| \leq 2^J$ and $|m| \leq 2^{J+1}$ we also have that $|y| \leq 2^{-j}3 \cdot 2^J$ so that we can estimate

$$\begin{aligned} |k^{\beta}_{j,m}(f)| &\leq c2^{jn}q'^{|\beta|}2^j \int\limits_{|y|\leq c_J 2^{-j}} |y|^{-\alpha}dy \\ &\leq cq'^{|\beta|}2^{j(\alpha+1)}. \end{aligned} \quad (4.14)$$

Together with (4.13) and (4.14) we obtain

$$\begin{aligned} (\|k(f)|l_p\|_s)^p &= \sum_{\beta,j,m} 2^{j(s-n/p)p}|k^{\beta}_{j,m}(f)|^p \\ &\leq \sum_{j=0}^{\infty} 2^{j(s-n/p)p} \left(\sum_{\beta,|m|\geq 2^{J+1}} |k^{\beta}_{j,m}(f)|^p + \sum_{\beta,|m|\leq 2^{J+1}} |k^{\beta}_{j,m}(f)|^p \right) \\ &\leq c\sum_{j=0}^{\infty} 2^{j(s-n/p)p}2^{j(\alpha+1)p} \left(\sum_{2^{j+1}\geq|m|\geq 2^{J+1}} (|m| - 2^J)^{(n-\alpha-1)p} + 2^{(J+1)n} \right). \end{aligned}$$

The last sum is bounded from above if $(n-\alpha-1)p < -n$ and we find the following sufficient condition

$$f \in B^s_p(\mathbb{R}^n), \quad \text{if} \quad s < \frac{n}{p} - \alpha - 1 < -n,$$

which corresponds for large p to the assertion (ii) of Lemma 1 of 2.3.1 in [19].

4.1.3 Local properties

Let Ω be a domain in $\mathbb{R}^n$, and $f, g \in S'(\mathbb{R}^n)$. Then we put

$$f = g \mod C^\infty \text{ in } \Omega, \tag{4.15}$$

if the restriction of $f - g$ to Ω is a C^∞-function. Let now f be given as in (4.11), that means

$$f = \sum_{\beta,j,m} k^\beta(2^{-j}, f)(2^{-j}m)\Phi^\beta_{j,m}.$$

Then we define for $x^0 \in \mathbb{R}^n$ and $K \in \mathbb{N}$ with $K \geq J$

$$f^{K,x^0}(x) = \sum_{\beta,j,m}^{K,x^0} k^\beta(2^{-j}, f)(2^{-j}m)\Phi^\beta(2^j x - m), \tag{4.16}$$

where the summation is restricted to all $j > J + K$ and m with

$$B_{x^0,2^{-K+1}} \cap B_{2^{-j}m,2^{-j}} \neq \emptyset \quad . \tag{4.17}$$

This condition ensures, that only those coefficients are taken into acount, that depend on information about the function f at most in the ball with radius 2^{-K+2} centered at x^0. This fact is the reason that the radius 2^{-K+2} appears in the definition of $s_{K,x}$ in 3.1, which will be used in section 6 again. Now we put

$$\|k(f)|l_p\|^{K,x^0}_s = \left(\sum_{\beta,j,m}^{K,x^0} 2^{j(s-n/p)p} |k^\beta(2^{-j}, f)(2^{-j}m)|^p \right)^{1/p} \tag{4.18}$$

with the same restrictions on the summation as in (4.16) and formulate the following relation between f and f^{K,x^0}.

Proposition 4.1 *Let $1 < p \leq \infty$, $s \leq t < 0$ and $f \in B^s_p(\mathbb{R}^n)$. Then*

$$f = f^{K,x^0} \mod C^\infty \text{ in } B_{x^0,2^{-K}} \tag{4.19}$$

and

$$\|f - f^{K,x^0} | B^t_p(B_{x^0,2^{-K}})\| \leq c 2^{K(t-s)} \|k(f)|l_p\|_s. \tag{4.20}$$

Furthermore let $s \leq \sigma < 0$. Then

$$\|k(f)|l_p\|_\sigma^{K,x^0} < \infty, \qquad \textit{implies} \qquad f^{K,x^0} \in B_p^\sigma(\mathbb{R}^n) \tag{4.21}$$

with

$$\|f^{K,x^0}|B_p^\sigma(\mathbb{R}^n)\| \leq c\|k(f)|l_p\|_\sigma^{K,x^0}, \tag{4.22}$$

and, conversely,

$$f^{K,x^0} \in B_p^\sigma(\mathbb{R}^n), \qquad \textit{implies} \qquad \|k(f)|l_p\|_\sigma^{K+2,x^0} < \infty \tag{4.23}$$

with

$$\|k(f)|l_p\|_\sigma^{K+2,x^0} \leq c\|f^{K,x^0}|B_p^\sigma(\mathbb{R}^n)\| + c2^{K(\sigma-s)}\|k(f)|l_p\|_s. \tag{4.24}$$

This is a modified version of Corollary 1 in [28]. In particular since (4.20), (4.22) and (4.24) are also to be taken into consideration, we shall expand the proof given there. This proposition shows that f^{K,x^0} is a local approximation of f around x^0 and its wavelet coefficients can be asked if the function f belongs locally to a Besov space with a higher degree of smoothness, see (4.21).

Proof *Step 1*

Let

$$\langle x \rangle = (1 + |x|^2)^{1/2} \qquad \text{for } x \in \mathbb{R}^n.$$

For any given $a > 0$ there are constants $C > 0$ and $c_a > 0$ such that

$$|D^\beta \omega^\vee(x)| \leq c_a 2^{C|\beta|} \langle x \rangle^{-a}, \quad x \in \mathbb{R}^n, \quad \beta \in \mathbb{N}_0^n,$$

where C is independent of x, a, β and c_a is independent of x, β, see [26]. Then by (4.6) it follows that

$$|(\omega^\beta)^\vee(y)| = \frac{i^{|\beta|} 2^{J|\beta|}}{(2\pi)^n \beta!} |D^\beta \omega^\vee(x)| \leq c 2^{-\varrho|\beta|} \langle y \rangle^{-a},$$

where both $\varrho > 0$ and $a > 0$ can be arbitrary chosen. Hence, by (4.8) and (4.9) we get

$$\begin{aligned} |D^\alpha \Phi^\beta(x)| &= \left| \sum_{m \in \mathbb{Z}^n} (\omega^\beta)^\vee(m) (D^\alpha \widehat{\varphi})(x+m) \right| \\ &\leq c 2^{-\varrho|\beta|} \sum_{m \in \mathbb{Z}^n} \langle m \rangle^{-a} \langle x+m \rangle^{-a}, \end{aligned}$$

because $D^\alpha \widehat{\varphi} \in S(\mathbb{R}^n)$. Now we can split the sum into the parts with $|m| \leq |x|/2$ and $|m| > |x|/2$ and obtain

$$|D^\alpha \Phi^\beta(x)| \leq c 2^{-\varrho|\beta|} \langle x \rangle^{-d}, \tag{4.25}$$

where both $\varrho > 0$ and $d > 0$ can be arbitrary chosen and c is independent of β and x.

Step 2

Let $f \in B^s_p(\mathbb{R}^n)$. We fix $\beta \in \mathbb{N}^n_0$ and $j \in \mathbb{N}_0$ in the sum (4.11) and denote the resulting sum over $m \in \mathbb{Z}^n$ by $f_{\beta,j}$. By (4.25) we have

$$\begin{aligned} |D^\alpha f_{\beta,j}(x)| &= \left| \sum_{m\in\mathbb{Z}^n} k^\beta_{j,m}(f)(D^\alpha \Phi^\beta)(2^j x - m) 2^{j|\alpha|} \right| \\ &\leq c 2^{-j(s-n/p)} 2^{j|\alpha|} 2^{-\varrho|\beta|} \left| \sum_{m\in\mathbb{Z}^n} 2^{j(s-n/p)} k^\beta_{j,m}(f) \langle 2^j x - m\rangle^{-d} \right| \\ &\leq c 2^{-j(s-n/p)} 2^{j|\alpha|} 2^{-\varrho|\beta|} \sup_m \{ 2^{j(s-n/p)} |k^\beta_{j,m}(f)| \} \sum_{m\in\mathbb{Z}^n} \langle 2^j x - m\rangle^{-d} \\ &\leq c 2^{-j(s-n/p)} 2^{j|\alpha|} 2^{-\varrho|\beta|} \|k(f)|l_p\|_s \sum_{m\in\mathbb{Z}^n} \langle 2^j x - m\rangle^{-d}, \end{aligned}$$

where both $d > 0$ and $\varrho > 0$ can be arbitrary chosen. The remaining sum over m is uniformly bounded for all $x \in \mathbb{R}^n$. Then it follows that $f_{\beta,j}$ and also $f_j = \sum_\beta f_{\beta,j}$ are C^∞-functions in $\mathbb{R}^n$.

Step 3

Now we prove (4.19). We assume $x_0 = 0$. By (4.11) and (4.16) we can write

$$f - f^{K,0} = \sum_{\beta,j,m}^{1} k^\beta_{j,m}(f) \Phi^\beta_{j,m} + \sum_{\beta,j,m}^{2} k^\beta_{j,m}(f) \Phi^\beta_{j,m}, \tag{4.26}$$

where in the first sum the summation is restricted to all $j > J + K$ and m with

$$|2^{-j} m| \geq 2^{-K+1} + 2^{-j} \tag{4.27}$$

and in the second sum the summation is restricted to all $j \leq J + K$. We begin with the first sum and have by (4.27)

$$|m| \geq 2^{j-K+1}. \tag{4.28}$$

Now, if we assume (4.28) and $|x| \leq 2^{-K}$, then $|m - 2^j x| \geq 2^{j-K}$ and we find similar to the end of Step 2 for all j, m and $|x| \leq 2^{-K}$

$$\left| D^\alpha \left(\sum_m^{1} k^\beta_{j,m}(f) \Phi^\beta_{j,m} \right)(x) \right| \leq c 2^{-j(s-n/p)} 2^{j|\alpha|} 2^{-\varrho|\beta|} 2^{(n-d)(j-K)} \|k(f)|l_p\|_s, \tag{4.29}$$

where both $d > n$ and $\varrho > 0$ can be arbitrary chosen. Now the same arguments as in Step 2 ensure that the first sum in (4.26) is a C^∞-function in the ball $B_{2^{-K+1}}$.

We can choose $\alpha = 0$ and d in (4.29) with $s - n/p + d - n > 0$, then with (2.10) we can state

$$\begin{aligned}\left\| \sum_{\beta,j,m}^{1} k_{j,m}^{\beta}(f)\Phi_{j,m}^{\beta} \middle| B_p^t(B_{2^{-K}}) \right\| &\le c2^{K(t-n/p)} \left\| \sum_{\beta,j,m}^{1} k_{j,m}^{\beta}(f)\Phi_{j,m}^{\beta} \middle| C(B_{2^{-K}}) \right\| \\ &\le c2^{K(t-n/p)} \sum_{j>J+K} 2^{-j(s-n/p+d-n)} 2^{K(d-n)} \|k(f)|l_p\|_s \\ &\le c2^{K(t-n/p)} 2^{-(J+K)(s-n/p)} 2^{-J(d-n)} \|k(f)|l_p\|_s \\ &\le c2^{K(t-s)} \|k(f)|l_p\|_s,\end{aligned}$$

where we included all constants that depend on J into the constant c. Because of Step 2, it is clear that the second sum in (4.26) is a C^∞-function. That already proves (4.19). But with (4.25) we can state further that

$$\begin{aligned}&\left\| \sum_{\beta,j,m}^{2} k_{j,m}^{\beta}(f)\Phi_{j,m}^{\beta} \middle| B_p^t(B_{2^{-K}}) \right\| \le c2^{K(t-n/p)} \left\| \sum_{\beta,j,m}^{2} k_{j,m}^{\beta}(f)\Phi_{j,m}^{\beta} \middle| C(B_{2^{-K}}) \right\| \\ &\le c2^{K(t-n/p)} \sum_{j=0}^{J+K} 2^{-j(s-n/p)} \sum_{\beta,m} 2^{j(s-n/p)} |k_{j,m}^{\beta}(f)\Phi_{j,m}^{\beta}| \\ &\le c2^{K(t-n/p)} \sum_{j=0}^{J+K} 2^{-j(s-n/p)} \sum_{\beta} 2^{-\varrho|\beta|} \sum_{m} 2^{j(s-n/p)} |k_{j,m}^{\beta}(f)| \langle 2^j x - m\rangle^{-d} \\ &\le c2^{K(t-n/p)} \sum_{j=0}^{J+K} 2^{-j(s-n/p)} \sum_{\beta} 2^{-\varrho|\beta|} \sup_m \{2^{j(s-n/p)} |k_{j,m}^{\beta}(f)|\} \sum_m \langle 2^j x - m\rangle^{-d} \\ &\le c2^{K(t-s)} \|k(f)|l_p\|_s,\end{aligned}$$

where we used (2.10) again. Together we obtain

$$\|f - f^{K,0}|B_p^t(B_{2^{-K}})\| \le c2^{K(t-s)} \|k(f)|l_p\|_s.$$

That proves (4.20).

Step 4

We prove (4.21) and (4.22). The function f^{K,x^0} is defined by

$$f^{K,x^0} = \sum_{\beta,j,m} \mu_{j,m}^{\beta} \Phi_{j,m}^{\beta}$$

with

$$\mu_{j,m}^{\beta} = \begin{cases} k_{j,m}^{\beta}(f) & : \ j > J + K,\ B_{x^0,2^{-K+1}} \cap B_{2^{-j}m,2^{-j}} \neq \emptyset, \\ 0 & : \ \text{otherwise.} \end{cases}$$

We know that

$$\|\mu|l_p\|_\sigma = \|k(f)|l_p\|_\sigma^{K,x^0} < \infty$$

because of our assumptions. Now Theorem 4.1 (i) gives $f^{K,x^0} \in B_p^\sigma(\mathbb{R}^n)$ and

$$\|f^{K,x^0}|B_p^\sigma(\mathbb{R}^n)\| \le c\|k(f)|l_p\|_\sigma^{K,x^0}.$$

<u>*Step 5*</u>

Now we prove (4.23) and (4.24). Let $f^{K,x^0} \in B_p^\sigma(\mathbb{R}^n)$, $s \le \sigma < 0$. By Theorem 4.1 (iii) we have

$$\|k(f^{K,x^0})|l_p\|_\sigma \le c\|f^{K,x^0}|B_p^\sigma(\mathbb{R}^n)\| < \infty. \tag{4.30}$$

Let again $x^0 = 0$. By (4.19) we have for $|x| \le 2^{-K-1}$

$$f(x) = f^{K,0}(x) + g(x) \quad \text{with} \quad g \in C_0^\infty(B_{2^{-K}}). \tag{4.31}$$

For the sequence-norm on the right-hand side of (4.23) we assume

$$B_{2^{-(K+2)+1}} \cap B_{2^{-j}m,2^{-j}} \neq \emptyset$$

for some $j > J + K + 2$. Then by (4.31) we have under this restriction

$$k_{j,m}^\beta(f) = k_{j,m}^\beta(f^{K,x^0}) + k_{j,m}^\beta(g). \tag{4.32}$$

If we write $g = f - f^{K,x^0}$ then by (4.32), (4.30) and (4.20) we get

$$\|k(f)|l_p\|_\sigma^{K+2,x^0} \le c\|f^{K,x^0}|B_p^\sigma(\mathbb{R}^n)\| + c2^{K(\sigma-s)}\|k(f)|l_p\|_s,$$

which is (4.24) and proves (4.23).

□

4.2 Wavelet frames for functions

Now we generalize the decomposition in 4.1 for $s \in \mathbb{R}$, originally stated in [28].

4.2.1 Definition and Theorem

To show the idea how to circumvent the restriction $s < 0$ in Theorem 4.1 we consider the operator

$$D_L = id + (-\Delta)^L, \quad L \in \mathbb{N}_0,$$

which maps any space $B_p^s(\mathbb{R}^n)$ with $s \in \mathbb{R}$ and $0 < p \le \infty$ isomorphically onto $B_p^{s-2L}(\mathbb{R}^n)$. Let $f \in B_p^s(\mathbb{R}^n)$ and $L \in \mathbb{N}_0$ with $s-2L < 0$. Then $D_L f \in B_p^{s-2L}(\mathbb{R}^n)$ and we have by (4.11)

$$f = \sum_{\beta,j,m} k_{j,m}^\beta(D_L f) D_L^{-1}[\Phi^\beta(2^j \cdot -m)](x).$$

Now we give the definitions for the resulting wavelets and coefficients.

Definition 4.2 *Let φ_0, φ and Ω^β be given by (2.1), (2.2) and (4.7). Let $\beta \in \mathbb{N}_0^n$ and $L \in \mathbb{N}_0$. Then the father L-wavelets $\Phi_F^{\beta,L}(x)$, the mother L-wavelets $\Phi_M^{\beta,L}(x)$ and the remainder L-wavelets $\Phi_j^{\beta,L}$ are given by*

$$
\begin{aligned}
(\Phi_F^{\beta,L})^\vee(\xi) &= \frac{\varphi_0(\xi)}{1+|\xi|^{2L}}\Omega^\beta(\xi), \quad \xi \in \mathbb{R}^n, \\
(\Phi_M^{\beta,L})^\vee(\xi) &= \frac{\varphi(\xi)}{|\xi|^{2L}}\Omega^\beta(\xi), \quad \xi \in \mathbb{R}^n, \\
(\Phi_j^{\beta,L})^\vee(\xi) &= -\frac{\varphi(\xi)}{|\xi|^{2L}(|\xi|^{2L}+2^{-2jL})}\Omega^\beta(\xi), \quad \xi \in \mathbb{R}^n,
\end{aligned}
$$

if $j \in \mathbb{N}$ and $\Phi_j^{\beta,L}(\xi) = 0$ if $j = 0$.

Remark 4.2 *All these wavelets are functions in $S(\mathbb{R}^n)$. For $L = 0$ we basically obtain the wavelets from Definition 4.1,*

$$\Phi_F^\beta = 2\Phi_f^{\beta,0}, \quad \Phi_M^\beta = \Phi_M^{\beta,0} = -2\Phi_j^{\beta,0},$$

where $j \in \mathbb{N}$ and $\beta \in \mathbb{N}_0^n$.

We generalize (4.10) by

$$\Phi^{\beta,L}(2^j x - m) = \begin{cases} \Phi_F^{\beta,L}(x-m) & : \ j = 0 \\ \Phi_M^{\beta,L}(2^j x - m) & : \ j \in \mathbb{N}. \end{cases}$$

Furthermore, we generalize the local means defined in (4.1)-(4.4). Let now

$$k_L^\beta(t,f)(x) = \int_{\mathbb{R}^n} k_L^\beta(y) f(x+ty)\,dy, \quad t > 0, \quad x \in \mathbb{R}^n$$

be the local means for $f \in S'(\mathbb{R}^n)$ with the kernel

$$k_L^\beta(x) = (-\Delta)^L k^\beta(x) \quad \text{for } L \in \mathbb{N}.$$

The corresponding norm $\|k_L(f)|l_p\|_s$ is defined as in (4.5). Then we can formulate the analogue to Theorem 4.1.

Theorem 4.2 *Let $1 < p \le \infty$, $s \in \mathbb{R}$ and $L \in \mathbb{N}_0$ with $s - 2L \le \tau$ for a negative number $\tau \le s$.*
(i) *Then $f \in B_p^\tau(\mathbb{R}^n)$ is an element of $B_p^s(\mathbb{R}^n)$ if, and only if, it can be represented as*

$$f = \sum_{\beta,j,m} \mu_{j,m}^\beta (\Phi^{\beta,L} + 2^{-2jL}\Phi_j^{\beta,L})(2^j x - m)$$

with $\|\mu|l_p\|_s < \infty$, with unconditional convergence in $B_p^\tau(\mathbb{R}^n)$. Furthermore,

$$\|f|B_p^s(\mathbb{R}^n)\| \sim \inf \|\mu|l_p\|_s,$$

where the infimum is taken over all admissible representations.
(ii) *Any $f \in B^s_p(\mathbb{R}^n)$ can be represented as*

$$f = \sum_{\beta,j,m} \mu^\beta_{j,m}(f)(\Phi^{\beta,L} + 2^{-2jL}\Phi^{\beta,L}_j)(2^j x - m) \tag{4.33}$$

with

$$\mu^\beta_{j,m}(f) = \left[k^\beta_L(2^{-j}, f) + 2^{-2jL}k^\beta(2^{-j}, f)\right](2^{-j}m), \tag{4.34}$$

unconditional convergence in $f \in B^s_p(\mathbb{R}^n)$. In addition we have

$$\|f|B^s_p(\mathbb{R}^n)\| \sim \|\mu(f)|l_p\|_s \sim \|k_L(f)|l_p\|_s + \|k(f)|l_p\|_\tau, \tag{4.35}$$

where the equivalence constants are independent of f.

This Theorem is basically the same as Theorem 3 in [28], only the formula (4.35) is different than in the original statement, so we prove only that.
Proof We start with a short calculation.

$$\begin{aligned} k^\beta(2^{-j}, D_L f)(x) &= k^\beta(2^{-j}, f)(x) + \int k^\beta(y)[(-\Delta)^L f](x + 2^{-j}y)dy \\ &= k^\beta(2^{-j}, f)(x) + 2^{2jL}k^\beta_L(2^{-j}, f)(x). \end{aligned}$$

Therefore, with (4.34) we have the equality

$$\mu^\beta_{j,m}(f) = 2^{-2jL}k^\beta_{j,m}(D_L f),$$

and Theorem 4.1 gives us

$$\|\mu(f)|l_p\|_s = \|k(D_L f)|l_p\|_{s-2L} \sim \|D_L f|B^{s-2L}_p(\mathbb{R}^n)\| \sim \|f|B^s_p(\mathbb{R}^n)\|.$$

Altogether we get

$$\begin{aligned} \|f|B^s_p(\mathbb{R}^n)\| &\leq c\|k_L(f)|l_p\|_s + c\|k(f)|l_p\|_{s-2L} \\ &\leq c\|k_L(f)|l_p\|_s + c\|k(f)|l_p\|_\tau. \end{aligned}$$

Conversely, by the above considerations we have also

$$\begin{aligned} \|k_L(f)|l_p\|_s + \|k(f)|l_p\|_\tau &\leq \|k(D_L f)|l_p\|_{s-2L} + c\|k(f)|l_p\|_\tau \\ &\leq c\|f|B^s_p(\mathbb{R}^n)\| + c\|k(f)|l_p\|_\tau \\ &\leq c\|f|B^s_p(\mathbb{R}^n)\|, \end{aligned}$$

which completes the proof.

□

4.2.2 Local properties

In order to use the previous Theorem to get local smoothness information, again we follow the same procedure as in 4.1.3. In analogy to (4.15), we write

$$f = g \mod B_p^\sigma \quad \text{in } \Omega$$

for a domain Ω in $\mathbb{R}^n$ and functions $f, g \in S'(\mathbb{R}^n)$, if the restriction of $f - g$ to Ω belongs to $B_p^\sigma(\Omega)$. Now we discuss which terms of the decomposition in (4.33) and (4.34) are important concerning local regularity.
Let for $f \in B_p^s(\mathbb{R}^n)$

$$\mu'^{\beta}_{j,m}(f) = 2^{-2jL} k^{\beta}_{j,m}(f),$$

then we have

$$\|\mu'(f)|l_p\|_{2L+\tau} = \|k(f)|l_p\|_\tau \leq c\|f|B_p^s(\mathbb{R}^n)\| \tag{4.36}$$

by Theorem 4.2. Let now f' be given in (4.33) with μ' in place of μ. Because (4.33) is a universal molecular representation in the sense of [9] (section 5), see also [28], we find

$$\|f'|B_p^{2L+\tau}(\mathbb{R}^n)\| \leq c\|\mu'(f)|l_p\|_{2L+\tau} \tag{4.37}$$

and therefore $f' \in B_p^{2L+\tau}(\mathbb{R}^n)$. Let also

$$\mu''^{\beta}_{j,m}(f) = 2^{-2jL} k_L^\beta(2^{-j}, f)(2^{-j}m),$$

then by Theorem 4.2 again we get

$$\|\mu''(f)|l_p\|_{s+2L} = \|k_L(f)|l_p\|_s \leq c\|f|B_p^s(\mathbb{R}^n)\| \tag{4.38}$$

and for

$$f''(x) = \sum_{\beta,j,m} 2^{-2jL} k_L^\beta(2^{-j}, f)(2^{-j}m)\Phi_j^{\beta,L}(2^j x - m),$$

which is also a universal molecular representation in the sense of [9], we have

$$\|f''|B_p^{s+2L}(\mathbb{R}^n)\| \leq c\|\mu''(f)|l_p\|_{s+2L}. \tag{4.39}$$

and therefore $f'' \in B_p^{s+2L}(\mathbb{R}^n)$. In other words we get for the remainder term

$$f(x) = \sum_{\beta,j,m} k_L^\beta(2^{-j}, f)(2^{-j}m)\Phi^{\beta,L}(2^j x - m) \mod B_p^{2L+\tau} \quad \text{in } \mathbb{R}^n. \tag{4.40}$$

Hence, the right-hand side of this equality is the main term of f as far as local regularity is concerned. In analogue to 4.1.3 let $x^0 \in \mathbb{R}^n$ and $K \in \mathbb{N}$ with $K \geq J$, then we define

$$f_L^{K,x^0}(x) = \sum_{\beta,j,m}^{K,x^0} \mu^\beta_{j,m}(f)(\Phi^{\beta,L} + 2^{-2jL}\Phi_j^{\beta,L})(2^j x - m)$$

and

$$\widetilde{f}_L^{K,x^0}(x) = \sum_{\beta,j,m}^{K,x^0} k_L^\beta(2^{-j},f)(2^{-j}m)\Phi^{\beta,L}(2^jx-m), \tag{4.41}$$

where the summation over j,m is always restricted as in (4.17). Let $\|k_L(f)|l_p\|_s^{K,x^0}$ be given by (4.18) with k_L in place of k, then we can formulate the analogue to Proposition 4.1.

Proposition 4.2 *Let $1 < p \leq \infty$, $s \in \mathbb{R}$ and $L \in \mathbb{N}_0$ with $s - 2L \leq \tau$ for a negative number $\tau \leq s$. Let $f \in B_p^s(\mathbb{R}^n)$ and f_L^{K,x^0}, $\widetilde{f}_L^{K,x^0}$ given as above. Then*

$$f = f_L^{K,x^0} \quad \mathrm{mod}\ C^\infty \quad \textit{in } B_{x^0,2^{-K}}$$

and

$$f = \widetilde{f}_L^{K,x^0} \quad \mathrm{mod}\ B_p^{2L+\tau} \quad \textit{in } B_{x^0,2^{-K}}$$

with

$$\|f-\widetilde{f}_L^{K,x^0}|B_p^t(B_{x^0,2^{-K}})\| \leq c2^{K(t-s)}\|k_L(f)|l_p\|_s + c2^{K(t-\tau)}\|k_L(f)|l_p\|_\tau + c\|k(f)|l_p\|_\tau \tag{4.42}$$

for $s \leq t < 2L+\tau$. Furthermore, let $s \leq \sigma < 2L+\tau$. Then

$$\|k_L(f)|l_p\|_\sigma^{K,x^0} < \infty, \quad \textit{implies} \quad \widetilde{f}_L^{K,x^0} \in B_p^\sigma(\mathbb{R}^n), \tag{4.43}$$

and, conversely,

$$\widetilde{f}_L^{K,x^0} \in B_p^\sigma(\mathbb{R}^n), \quad \textit{implies} \quad \|k_L(f)|l_p\|_\sigma^{K+2,x^0} < \infty.$$

This is again a slightly modified version of the corresponding Corollary 2 in [28]. The added line (4.42) will be proved below but the rest of the proof is analogous to Proposition 4.1 with some modifications according to the previous discussion. We want to emphasize that because (4.41) is a universal molecular representation in the sense of [9] again we get the following estimate

$$\|\widetilde{f}_L^{K,x^0}|B_p^\sigma(\mathbb{R}^n)\| \leq c\|k_L(f)|l_p\|_\sigma^{K,x^0}, \tag{4.44}$$

which already proves (4.43).

Proof By using the above notation we have by triangle inequality

$$\|f-\widetilde{f}_L^{K,x^0}|B_p^t(B_{x^0,2^{-K}})\| \leq \|f'|B_p^t(\mathbb{R}^n)\| + \|f''|B_p^t(\mathbb{R}^n)\| + \|\widetilde{f}_L - \widetilde{f}_L^{K,x^0}|B_p^t(B_{x^0,2^{-K}})\|, \tag{4.45}$$

where

$$\widetilde{f}_L = \sum_{\beta,j,m} k_L^\beta(2^{-j},f)(2^{-j}m)\Phi^{\beta,L}(2^jx-m)$$

is the right-hand side of (4.40). For the norm of f' we know by embedding and (4.36), (4.37)

$$\|f'|B^t_p(\mathbb{R}^n)\| \le c\|f'|B^{2L+\tau}_p\| \le c\|k(f)|l_p\|_\tau \tag{4.46}$$

and for the norm of f'' also by embedding and (4.38), (4.39)

$$\|f''|B^t_p(\mathbb{R}^n)\| \le c\|f''|B^{2L+s}_p\| \le c\|k_L(f)|l_p\|_s. \tag{4.47}$$

Now we take care about the third term on the right-hand side of (4.45) by using basically the same arguments as in the steps 1-3 of the proof of Proposition 4.1. At first we write in analogy to (4.26) with $x^0 = 0$

$$\begin{aligned}\widetilde{f}_L - \widetilde{f}^{K,0}_L = \quad & \sum_{\beta,j,m}{}^{1} k^\beta_L(2^{-j}, f)(2^{-j}m)\Phi^{\beta,L}(2^jx - m) \\ + \quad & \sum_{\beta,j,m}{}^{2} k^\beta_L(2^{-j}, f)(2^{-j}m)\Phi^{\beta,L}(2^jx - m),\end{aligned} \tag{4.48}$$

where in the first sum the summation is restricted to all $j > J + K$ and m with

$$|2^{-j}m| \ge 2^{-K+1} + 2^{-j} \tag{4.49}$$

and in the second sum the summation is restricted to all $j \le J + K$. It is easy to see that the analogue to formula (4.25)

$$|D^\alpha\Phi^{\beta,L}(x)| \le c2^{-\varrho|\beta|}\langle x\rangle^{-d}$$

holds for arbitrary chosen numbers $\varrho, d > 0$. Then we can calculate in the same way as in step 2 of the proof of Proposition 4.1

$$\begin{aligned}&\left|\sum_m k^\beta_L(2^{-j}, f)(2^{-j}m)(D^\alpha\Phi^{\beta,L})(2^jx - m)2^{j|\alpha|}\right| \\ &\qquad \le \quad c2^{-j(s-n/p)}2^{j|\alpha|}2^{-\varrho|\beta|}\|k_L(f)|l_p\|_s \sum_{m\in\mathbb{Z}^n}\langle 2^jx - m\rangle^{-d},\end{aligned}$$

With the restrictions (4.49) for the first sum in (4.48) we get for $|x| \le 2^{-K}$

$$\begin{aligned}&\left|\sum_{\beta,j,m}{}^{1} k^\beta_L(2^{-j}, f)(2^{-j}m)(D^\alpha\Phi^{\beta,L})(2^jx - m)2^{j|\alpha|}\right| \\ &\qquad \le \quad c\sum_{\beta,j>J+K} 2^{-j(s-n/p+d-n)}2^{j|\alpha|}2^{-\varrho|\beta|}2^{K(d-n)}\|k_L(f)|l_p\|_s\end{aligned} \tag{4.50}$$

with arbitrary chosen $d > n$ and $\varrho > 0$ in analogy to (4.29). For the second sum in (4.48) we find in a similar way

$$\begin{aligned}&\left|\sum_{\beta,j,m}{}^{2} k^\beta_L(2^{-j}, f)(2^{-j}m)(D^\alpha\Phi^{\beta,L})(2^jx - m)2^{j|\alpha|}\right| \\ &\qquad \le \quad c\sum_{\beta,j\le J+K} 2^{-j(r-n/p)}2^{j|\alpha|}2^{-\varrho|\beta|}\|k_L(f)|l_p\|_r\end{aligned} \tag{4.51}$$

for an arbitrary real number r. After these calculations we estimate the third term of (4.45). We start with (2.13) and get by embedding

$$\begin{aligned}\|\widetilde{f}_L - \widetilde{f}_L^{K,x^0}|B_p^t(B_{x^0,2^{-K}})\| &\leq c2^{K(t-n/p)}\|(\widetilde{f}_L - \widetilde{f}_L^{K,x^0})(2^{-K}\cdot)|B_p^t(B_{x^0,1})\| \\ &\leq c2^{K(t-n/p)}\|(\widetilde{f}_L - \widetilde{f}_L^{K,x^0})(2^{-K}\cdot)|C^{2L}(B_{x^0,1})\|. \quad (4.52)\end{aligned}$$

We shall use the equivalent norm

$$\|g|C^{2L}(\Omega)\| \leq c\|g|C(\Omega)\| + c\sum_{|\alpha|=2L}\|D^\alpha g|C(\Omega)\|. \qquad (4.53)$$

Then we can estimate by (4.50) with sufficiently large d and (4.51) with $r = \tau$

$$\begin{aligned}&\|(\widetilde{f}_L - \widetilde{f}_L^{K,x^0})(2^{-K}\cdot)|C(B_{x^0,1})\| = \|\widetilde{f}_L - \widetilde{f}_L^{K,x^0}|C(B_{x^0,2^{-K}})\| \\ &\quad\leq c\sum_{\beta,j>J+K} 2^{-j(s-n/p+d-n)}2^{-\varrho|\beta|}2^{K(d-n)}\|k_L(f)|l_p\|_s \\ &\quad\quad + c\sum_{\beta,j\leq J+K} 2^{-j(\tau-n/p)}2^{-\varrho|\beta|}\|k_L(f)|l_p\|_\tau \\ &\quad\leq c2^{K(d-n)}2^{-K(s-n/p+d-n)}\|k_L(f)|l_p\|_s + c2^{-K(\tau-n/p)}\|k_L(f)|l_p\|_\tau \\ &\quad\leq c2^{-K(s-n/p)}\|k_L(f)|l_p\|_s + c2^{-K(\tau-n/p)}\|k_L(f)|l_p\|_\tau. \qquad (4.54)\end{aligned}$$

For the derivatives we find by (4.50) with sufficiently large d and (4.51) with $r = s$

$$\begin{aligned}&\sum_{|\alpha|=2L}\|D^\alpha[(\widetilde{f}_L - \widetilde{f}_L^{K,x^0})(2^{-K}\cdot)]|C(B_{x^0,1})\| \\ &= \sum_{|\alpha|=2L} 2^{-K|\alpha|}\|[D^\alpha(\widetilde{f}_L - \widetilde{f}_L^{K,x^0})](2^{-K}\cdot)|C(B_{x^0,1})\| \\ &= 2^{-2LK}\sum_{|\alpha|=2L}\|[D^\alpha(\widetilde{f}_L - \widetilde{f}_L^{K,x^0})]|C(B_{x^0,2^{-K}})\| \\ &\leq c2^{-2LK}\sum_{|\alpha|=2L}\sum_{\beta,j>J+K} 2^{-j(s-n/p+d-n-|\alpha|)}2^{K(d-n)}2^{-\varrho|\beta|}\|k_L(f)|l_p\|_s \\ &\quad + c2^{-2LK}\sum_{|\alpha|=2L}\sum_{\beta,j\leq J+K} 2^{-j(s-n/p-|\alpha|)}2^{-\varrho|\beta|}\|k_L(f)|l_p\|_s \\ &\leq c2^{-2LK}\left(2^{K(d-n)}2^{-K(s-n/p+d-n-2L)}\|k_L(f)|l_p\|_s + c2^{-K(s-n/p-2L)}\|k_L(f)|l_p\|_s\right) \\ &\leq c2^{-K(s-n/p)}\|k_L(f)|l_p\|_s \qquad (4.55)\end{aligned}$$

Inserting (4.54) and (4.55) with (4.53) into (4.52) we find

$$\|\widetilde{f}_L - \widetilde{f}_L^{K,x^0}|B_p^t(B_{x^0,2^{-K}})\| \leq c2^{K(t-s)}\|k_L(f)|l_p\|_s + c2^{K(t-\tau)}\|k_L(f)|l_p\|_\tau$$

which together with (4.46), (4.47) and (4.45) proves the desired estimate.

□

5 Decomposition with C^r-wavelets

In contrast to section 4 we discuss now a wavelet decomposition where the building blocks are C^r-functions but have a compact support. This section is essentially based on [29].

5.1 Definition and Theorem

Let $L_j = L = 2^n - 1$ if $j \in \mathbb{N}$ and $L_0 = 1$. Then for any number $r \in \mathbb{N}$ there are functions $\psi_0(x) \in C^r(\mathbb{R}^n)$ and $\psi^l(x) \in C^r(\mathbb{R}^n)$, $l = 1, \ldots, L$, with

$$\operatorname{supp} \psi_0(x),\ \operatorname{supp} \psi^l(x) \subset B_{2^{\tilde{J}}} \tag{5.1}$$

for a $\tilde{J} \in \mathbb{N}$ and

$$\int_{\mathbb{R}^n} x^\alpha \psi^l(x) dx = 0, \qquad \text{for } \alpha \in \mathbb{N}_0^n, |\alpha| \le r,$$

such that

$$\{2^{jn/2} \psi^l_{j,m}(x) : j \in \mathbb{N}_0, 1 \le l \le L_j, m \in \mathbb{Z}^n\}$$

with

$$\psi^l_{j,m}(x) = \begin{cases} \psi_0(x-m) & : \ j = 0, m \in \mathbb{Z}^n, l = 1, \\ \psi^l(2^{j-1}x - m) & : \ j \in \mathbb{N}, m \in \mathbb{Z}^n, 1 \le l \le L, \end{cases}$$

is an orthonormal basis in $L_2(\mathbb{R}^n)$.
The original version of such a system goes back to I. Daubechies, see [7]. For a detailed description we refer to [17] and [35].
We want to give the counterpart to Theorem 4.1 with compactly supported wavelets. For that purpose we need suitable sequence spaces.

Definition 5.1 *Let $s \in \mathbb{R}$ and $0 < p \le \infty$. Then the space b^s_p consists of all sequences*

$$\lambda = \{\lambda^l_{j,m} \in \mathbb{C} : j \in \mathbb{N}_0, 1 \le l \le L_j, m \in \mathbb{Z}^n\}$$

for which the quasi-norm

$$\|\lambda | b^s_p\| = \left(\sum_{l,j,m} 2^{j(s-n/p)p} |\lambda^l_{j,m}|^p \right)^{1/p} \tag{5.2}$$

(with the usual mofification for $p = \infty$) is finite.

The following Theorem was published by Triebel in [29], the proof is also given there.

Theorem 5.1 *Let s and p given as above. Then there is a natural number $r(s,p)$ such that for all $r > r(s,p)$ the following is true:*
Let $f \in S'(\mathbb{R}^n)$, then $f \in B^s_p(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{l,j,m} \lambda^l_{j,m}(f)\psi^l_{j,m} \qquad \text{with } \|\lambda|b^s_p\| < \infty, \tag{5.3}$$

with unconditional convergence in $S'(\mathbb{R}^n)$. Furthermore the representation (5.3) *is unique, $\lambda^l_{j,m}(f) = 2^{jn}(f, \psi^l_{j,m})$, and*

$$\|f|B^s_p(\mathbb{R}^n)\| \sim \|\lambda(f)|b^s_p\|$$

in the sense of equivalent quasi-norms.
In addition, for $p < \infty$, (5.3) *converges unconditionally in $B^s_p(\mathbb{R}^n)$ and $\{\psi^l_{j,m}\}$ is an unconditional Schauder basis in $B^s_p(\mathbb{R}^n)$.*

Remark 5.1 *In [29], Corollary 5, was proved, that in the Theorem one can choose*

$$r(s,p) = \max(s, \frac{2n}{p} + \frac{n}{2} - s).$$

This result allows us again to extract local regularity assertions.

5.2 Local properties

We follow the same idea as for the non-compactly supported wavelets. We define for $x^0 \in \mathbb{R}^n$ and $K \in \mathbb{N}$ with $K \geq \tilde{J}$

$$f_{K,x^0} = \sum_{l,j,m}^{K,x^0} \lambda^l_{j,m}(f)\psi^l_{j,m},$$

where the summation is restricted to all $j > \tilde{J} + K$ and m with

$$B_{x^0,2^{-K+1}} \cap B_{2^{-j}m,2^{-j}} \neq \emptyset \quad .$$

The corresponding norm is given by

$$\|\lambda(f)|b^s_p\|^{K,x^0} = \left(\sum_{l,j,m}^{K,x^0} 2^{j(s-n/p)p} |\lambda^l_{j,m}(f)|^p \right)^{1/p} .$$

Now we can formulate a Proposition which is in some sense the analogue to Proposition 4.1.

Proposition 5.1 *Let $1 < p \leq \infty$, $s < n/p$, $r > r(t,p)$ and $f \in B^s_p(\mathbb{R}^n)$. Then for $s \leq t$*

$$f = f_{K,x^0} \mod C^r \text{ in } B_{x^0,2^{-K}} \tag{5.4}$$

and

$$\|f - f_{K,x^0}|B^t_p(B_{x^0,2^{-K}})\| \leq c2^{K(t-s)}\|\lambda|b^s_p\|. \tag{5.5}$$

Furthermore let $s \leq \sigma$. Then

$$\|\lambda(f)|b^\sigma_p\|^{K,x^0} < \infty \quad \text{implies} \quad f_{K,x^0} \in B^\sigma_p(\mathbb{R}^n) \tag{5.6}$$

with

$$\|f_{K,x^0}|B^\sigma_p(\mathbb{R}^n)\| \leq c\|\lambda(f)|b^\sigma_p\|^{K,x^0}. \tag{5.7}$$

Proof We prove (5.4). We assume $x^0 = 0$ and write

$$f - f_{K,0} = \sum_{l,j,m}^{1} \lambda^l_{j,m}\psi^l_{j,m} + \sum_{l,j,m}^{2} \lambda^l_{j,m}\psi^l_{j,m},$$

where in the first sum the summation is restricted to all $j > \tilde{J} + K$ and m with

$$|2^{-j}m| \geq 2^{-K+1} + 2^{-j} \tag{5.8}$$

and in the second sum the summation is restricted to all $j \leq \tilde{J} + K$. Now we estimate

$$\begin{aligned}&\|f - f_{K,0}|C^r(B_{0,2^{-K}})\| \\ &\quad\leq \sum_{|\alpha|\leq r}\left(\sup_{|x|\leq 2^{-K}} \sum_{l,j,m}^{1} |\lambda^l_{j,m}D^\alpha\psi^l_{j,m}(x)| + \sup_{|x|\leq 2^{-K}} \sum_{l,j,m}^{2} |\lambda^l_{j,m}D^\alpha\psi^l_{j,m}(x)|\right).\end{aligned} \tag{5.9}$$

Because of (5.1) we know that

$$|\psi^l(2^jx - m)| = 0 \quad \text{if} \quad |2^jx - m| > 2^{\tilde{J}}.$$

The supremum is taken over $|x| \leq 2^{-K}$, this means

$$|\psi^l(2^jx - m)| = 0 \quad \text{if} \quad |2^{-j}m| > 2^{-K} + 2^{\tilde{J}-j}. \tag{5.10}$$

But the sum with superscript 1 in (5.9) fulfills this condition for the summation over m by (5.8), therefore this sum vanishes. Furthermore we can estimate

$$\begin{aligned}\|f - f_{K,0}|C^r(B_{2^{-K}})\| &\leq \sup_{|x|\leq 2^{-K}} \sum_{l,j,m}^{2} |\lambda^l_{j,m}| \sum_{|\alpha|\leq r} |D^\alpha\psi^l_{j,m}(x)| \\ &\leq \sum_{l,j\leq\tilde{J}+K} 2^{-j(s-n/p)} \sum_m 2^{j(s-n/p)}|\lambda^l_{j,m}| \sup_{|x|\leq 2^{-K}} \sum_{|\alpha|\leq r} 2^{j|\alpha|}|(D^\alpha\psi^l)(2^jx - m)| \\ &\leq c \sum_{l,j\leq\tilde{J}+K} 2^{-j(s-n/p)}\|\lambda|b^s_p\| 2^{jr}\|\psi^l|C^r(\mathbb{R}^n)\| \\ &\leq c2^{-(\tilde{J}+K)(s-r-n/p)}\|\lambda|b^s_p\|.\end{aligned} \tag{5.11}$$

That proves (5.4). To prove (5.5) we use formula (2.13) and (5.11) to obtain

$$
\begin{aligned}
\|f - f_{K,0}|B^t_p(B_{2^{-K}})\| &\le c2^{K(t-n/p)}\|(f - f_{K,0})(2^{-K}\cdot)|B^t_p(B_1)\| \\
&\le c2^{K(t-n/p)}\|(f - f_{K,0})(2^{-K}\cdot)|C^r(B_1)\| \\
&\le c2^{K(t-n/p)}\Big(\|(f - f_{K,0})(2^{-K}\cdot)|C(B_1)\| \\
&\qquad + \sum_{|\alpha|=r}\|D^\alpha[(f - f_{K,0})(2^{-K}\cdot)]|C(B_1)\|\Big) \\
&\le c2^{K(t-n/p)}\Big(\|f - f_{K,0}|C(B_{2^{-K}})\| \\
&\qquad + \sum_{|\alpha|=r}2^{-|\alpha|K}\|D^\alpha(f - f_{K,0})|C(B_{2^{-K}})\|\Big) \\
&\le c2^{K(t-n/p)}\|f - f_{K,0}|C(B_{2^{-K}})\| + c2^{K(t-r-n/p)}\|f - f_{K,0}|C^r(B_{2^{-K}})\| \\
&\le c2^{K(t-n/p)}2^{-(\tilde{J}+K)(s-n/p)}\|\lambda|b^s_p\| + c2^{K(t-r-n/p)}2^{-(\tilde{J}+K)(s-r-n/p)}\|\lambda|b^s_p\| \\
&\le c2^{K(t-s)}\|\lambda|b^s_p\|.
\end{aligned}
$$

Now we prove (5.6) and (5.7). By definition we know

$$f_{K,x^0} = \sum_{l,j,m} C^l_{j,m}(f)\psi^l_{j,m}$$

with

$$C^l_{j,m}(f_{K,x^0}) = \begin{cases} \lambda^l_{j,m}(f) & : \quad j > \tilde{J}+K,\ B_{x^0,2^{-K+1}} \cap B_{2^{-j}m,2^{-j}} \neq \emptyset \\ 0 & : \quad \text{otherwise.} \end{cases}$$

In addition we have

$$\|C|b^\sigma_p\| = \|\lambda(f)|b^\sigma_p\|^{K,x^0} < \infty$$

by assumption. Now Theorem 5.1 gives $f_{K,x^0} \in B^\sigma_p(\mathbb{R}^n)$ and

$$\|f_{K,x^0}|B^\sigma_p(\mathbb{R}^n)\| \le c\|\lambda(f)|b^\sigma_p\|^{K,x^0}.$$

□

6 Main results

Here we formulate different possibilities to characterize the space $B^{\mathbb{S},s_0}_p(\mathbb{R}^n)$ by wavelet coefficients of its elements. This enables us to calculate a few more examples explicitely, which will be done in 6.2, and to prove the relation between $B^{\mathbb{S},s_0}_p(\mathbb{R}^n)$ and the so-called Two-microlocal spaces in 6.3. Furthermore, the results given here are essential to investigate some special problems as we will see in section 7.

6.1 Theorems

We start with the case of C^∞-wavelets from section 4.

Theorem 6.1 *Let $1 < p \leq \infty$ and let $\mathbb{S}$ be a negative lower semi-continuous function in $\mathbb{R}^n$ that is bounded from below. Then there are two constants $c_1, c_2 > 0$ such that for all $f \in B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$,*

$$\begin{aligned} & c_1 \|k(f)|l_p\|_{s_0} + c_1 \sup_{K \geq J, x \in \mathbb{R}^n} 2^{-K(s_{K,x}-s_0)} \|k(f)|l_p\|_{s_{K,x}}^{K+2,x} \\ \leq \quad & \|f|B_p^{s_0}(\mathbb{R}^n)\| + \sup_{K \geq J, x \in \mathbb{R}^n} 2^{-K(s_{K,x}-s_0)} \|f|B_p^{s_{K,x}}(B_{x,2^{-K}})\| \qquad (6.1) \\ \leq \quad & c_2 \|k(f)|l_p\|_{s_0} + c_2 \sup_{K \geq J, x \in \mathbb{R}^n} 2^{-K(s_{K,x}-s_0)} \|k(f)|l_p\|_{s_{K,x}}^{K,x}. \end{aligned}$$

Remark 6.1 *Here an unavoidable index-shifting from K to $K+2$ appears. Otherwise we would have an equivalent quasi-norm for our space $B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$.*

A first proof of Theorem 6.1 is given by Triebel in [27] but we shall give a shorter one here.

Proof We only have to take care about the supremum terms because Theorem 4.1 gives the equivalence of the first terms on each side.

Step 1

We start with the left-hand side of (6.1). Let $f \in B_p^{s_0}(\mathbb{R}^n)$ and

$$g \in B_p^{s_{K,x}}(\mathbb{R}^n) \quad \text{with} \quad g|B_{x,2^{-K}} = f|B_{x,2^{-K}}. \qquad (6.2)$$

Then for all coefficients in the norm

$$\|k(f)|l_p\|_{s_{K,x}}^{K+2,x} \quad \text{we have} \quad k_{j,m}^\beta(g) = k_{j,m}^\beta(f).$$

Therefore, we have

$$\begin{aligned} \|k(f)|l_p\|_{s_{K,x}}^{K+2,x} &= \|k(g)|l_p\|_{s_{K,x}}^{K+2,x} \\ &\leq \|k(g)|l_p\|_{s_{K,x}} \leq c\|g|B_p^{s_{K,x}}(\mathbb{R}^n)\|, \end{aligned}$$

by Theorem 4.1. But because this inequality holds for all g with (6.2), it also holds for the infimum over all such g and so by Definition 2.3 we get

$$\|k(f)|l_p\|_{s_{K,x}}^{K+2,x} \leq c\|f|B_p^{s_{K,x}}(B_{x,2^{-K}})\|.$$

Step 2

Now we prove the right-hand side of (6.1). We write $f = (f - f^{K,x}) + f^{K,x}$, then the triangle inequality and the formulas (4.20), (4.22) lead to

$$
\begin{aligned}
&2^{-K(s_{K,x}-s_0)}\|f|B_p^{s_{K,x}}(B_{x,2^{-K}})\| \\
&\quad\leq\; 2^{-K(s_{K,x}-s_0)}\left(\|f-f^{K,x}|B_p^{s_{K,x}}(B_{x,2^{-K}})\|+\|f^{K,x}|B_p^{s_{K,x}}(B_{x,2^{-K}})\|\right) \\
&\quad\leq\; 2^{-K(s_{K,x}-s_0)}\left(c2^{K(s_{K,x}-s_0)}\|k(f)|l_p\|_{s_0}+c\|k(f)|l_p\|_{s_{K,x}}^{K,x}\right) \\
&\quad\leq\; c2^{-K(s_{K,x}-s_0)}\|k(f)|l_p\|_{s_{K,x}}^{K,x}+c\|k(f)|l_p\|_{s_0},
\end{aligned}
$$

which is the desired estimate.

□

Now we can state a similar result for the generalized decomposition with C^∞-wavelets from 4.2.

Theorem 6.2 *Let $1 < p \leq \infty$ and let $\mathbb{S} : x \mapsto s(x)$ be a bounded semi-continuous function in $\mathbb{R}^n$ with $s_{\max} - 2L \leq s_0 < 0$ for a $L \in \mathbb{N}_0$. Then for all $f \in B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$,*

$$
\begin{aligned}
&c\|k_L(f)|l_p\|_{s_0}+c\|k(f)|l_p\|_{s_0}+c\sup_{K\geq J,x\in\mathbb{R}^n}2^{-K(s_{K,x}-s_0)}\|k_L(f)|l_p\|_{s_{K,x}}^{K+2,x} \\
\leq\;& \|f|B_p^{\mathbb{S},s_0}(\mathbb{R}^n)\| \qquad (6.3)\\
\leq\;& c\|k_L(f)|l_p\|_{s_0}+c\|k(f)|l_p\|_{s_0}+c\sup_{K\geq J,x\in\mathbb{R}^n}2^{-K(s_{K,x}-s_0)}\|k_L(f)|l_p\|_{s_{K,x}}^{K,x}.
\end{aligned}
$$

Proof By Theorem 4.2, formula (4.35) with $\tau = s_0$, we know

$$\|f|B_p^{s_0}(\mathbb{R}^n)\| \sim \|k_L(f)|l_p\|_{s_0}+\|k(f)|l_p\|_{s_0},$$

which means we only have to care about the suprema.
Step 1
The proof of the left hand-side of (6.3) is the same as in the proof of the last Theorem using Theorem 4.2 instead of Theorem 4.1.
Step 2
For the right-hand side of (6.3) we write $f = (f - \widetilde{f}_L^{K,x}) + \widetilde{f}_L^{K,x}$. Then the triangle inequality and the formulas (4.42) with $t = s_{K,x}$ and $s = \tau = s_0$ and (4.44) lead to

$$
\begin{aligned}
&2^{-K(s_{K,x}-s_0)}\|f|B_p^{s_{K,x}}(B_{x,2^{-K}})\| \\
&\quad\leq\; 2^{-K(s_{K,x}-s_0)}\left(\|f-\widetilde{f}_L^{K,x}|B_p^{s_{K,x}}(B_{x,2^{-K}})\|+\|\widetilde{f}_L^{K,x}|B_p^{s_{K,x}}(B_{x,2^{-K}})\|\right) \\
&\quad\leq\; 2^{-K(s_{K,x}-s_0)}\left(c2^{K(s_{K,x}-s_0)}\|k_L(f)|l_p\|_{s_0}+c\|k(f)|l_p\|_{s_0}+c\|k_L(f)|l_p\|_{s_{K,x}}^{K,x}\right) \\
&\quad\leq\; c2^{-K(s_{K,x}-s_0)}\|k_L(f)|l_p\|_{s_{K,x}}^{K,x}+c\|k_L(f)|l_p\|_{s_0}+c\|k(f)|l_p\|_{s_0},
\end{aligned}
$$

where we used $2^{-K(s_{K,x}-s_0)} \leq 1$. That completes the proof.

□

Now we will state the analog Theorem for the case of compactly supported wavelets from section 5.

Theorem 6.3 *Let $1 < p \leq \infty$ and let $\mathbb{S}$ be a bounded lower semi-continuous function in $\mathbb{R}^n$. Then there are two constants $c_1, c_2 > 0$ such that for all functions $f \in B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$ with $s_0 < 0$,*

$$\begin{aligned} & c_1 \|\lambda(f)|b_p^{s_0}\| + c_1 \sup_{K \geq \tilde{J}, x \in \mathbb{R}^n} 2^{-K(s_{K,x}-s_0)} \|\lambda(f)|b_p^{s_{K,x}}\|^{K+2,x} \\ \leq \;& \|f|B_p^{s_0}(\mathbb{R}^n)\| + \sup_{K \geq \tilde{J}, x \in \mathbb{R}^n} 2^{-K(s_{K,x}-s_0)} \|f|B_p^{s_{K,x}}(B_{x,2^{-K}})\| \\ \leq \;& c_2 \|\lambda(f)|b_p^{s_0}\| + c_2 \sup_{K \geq \tilde{J}, x \in \mathbb{R}^n} 2^{-K(s_{K,x}-s_0)} \|\lambda(f)|b_p^{s_{K,x}}\|^{K,x}. \end{aligned} \tag{6.4}$$

The proof works in exactly the same way as for Theorem 6.1, using Theorem 5.1 and the estimates from Proposition 5.1.

Remark 6.2 *The assumption $f \in B_p^{\mathbb{S},s_0}(\mathbb{R}^n)$ in the last three Theorems is not really necessary. The assertions remain true if we assume $f \in S'(\mathbb{R}^n)$.*

These results show that in order to treat problems concerning local smoothness behavior of a function it is enough to have information about their wavelet coefficients. Especially Theorem 6.3 gives a uniform assertion without restrictions on the smoothness function $s(x)$ and is used as the main tool in 6.3.

6.2 Examples

Now we calculate three examples to see the usefulness of such norm estimates. We start with $f = \delta$ and verify Example 3.1 again.

Example 6.1 *Let $\mathbb{S} : x \mapsto s(x)$ be a negative lower semi-continuous function in $\mathbb{R}^n$ that is bounded from below with $s(0) = n/p - n - \varepsilon$ for $\varepsilon > 0$. Then*

$$\delta \in B_p^{\mathbb{S},s_{\min}}(\mathbb{R}^n).$$

Proof For the first term on the right-hand side of (6.1) we have with (4.12)

$$\begin{aligned} \|k(\delta)|l_p\|_{s_{\min}} & \leq \|k(\delta)|l_p\|_{s(0)} \\ & \leq c \left(\sum_{j=0}^{\infty} 2^{j(s(0)-n/p)p} 2^{jnp} \right)^{1/p} < \infty. \end{aligned}$$

However, the second term is more interesting

$$\|k(\delta)|l_p\|_{s_{K,x}}^{K,x} = \left(\sum_{\beta,j,m}^{K,x} 2^{j(s_{K,x}-n/p)p} 2^{jnp} |(2^{-J}m)^\beta|^p k(-m)^p \right)^{1/p}.$$

Again we put $q = \sup_{m \in \operatorname{supp} k} |2^{-J} m|$, then $q < 1$ holds and we see that

$$\|k(\delta)|l_p\|^{K,x}_{s_{K,x}} \le c \left(\sum_{j,m}^{K,x} 2^{j(s_{K,x}-n/p)p} 2^{jnp} k(-m)^p \right)^{1/p}.$$

In the case $0 \notin B_{x,2^{-K+2}}$, we have by $j > J + K$, $K \ge J$ and $|m| < 2^J$

$$|x - 2^{-j} m| \ge 2^{-K+2} - 2^{-K} \ge 2^{-K+1} + 2^{-j}$$

and it follows that

$$B_{x,2^{-K+1}} \cap B_{2^{-j}m,2^{-j}} = \emptyset.$$

Therefore, we can write

$$\sup_{K,x} 2^{-K(s_{K,x}-s_{\min})} \|k(\delta)|l_p\|^{K,x}_{s_{K,x}} = \sup_{\substack{K,x \\ 0 \in B_{x,2^{-K+2}}}} 2^{-K(s_{K,x}-s_{\min})} \|k(\delta)|l_p\|^{K,x}_{s_{K,x}}.$$

For all K, x with $0 \in B_{x,2^{-K+2}}$ the relation $s_{K,x} \le s(0)$ holds, hence, with $|m| < 2^J$ we can estimate

$$\sup_{K,x} 2^{-K(s_{K,x}-s_{\min})} \|k(\delta)|l_p\|^{K,x}_{s_{K,x}} \le c \left(\sum_{j=2J+1}^{\infty} 2^{j(s(0)-n/p)p} 2^{jnp} \right)^{1/p} < \infty,$$

where we used $2^{-K(s_{K,x}-s_{\min})} < 1$. That proves the assertion.

□

Example 6.2 *Let Γ be a d-set and μ the corresponding Radon measure in $\mathbb{R}^n$ with $0 < d < n$, hence,*

$$\mu(B_{\gamma,r}) \sim r^d \quad \text{if} \quad \gamma \in \Gamma = \operatorname{supp} \mu. \tag{6.5}$$

Let $\mathbb{S} : x \mapsto s(x)$ be a negative lower semi-continuous function in $\mathbb{R}^n$ that is bounded from below with

$$s(x) = -\frac{n-d}{p'} - \varepsilon \quad \text{if} \quad x \in \Gamma,$$

$\varepsilon > 0$ and $1 = 1/p + 1/p'$. Then

$$\mu \in B_p^{\mathbb{S},s_{\min}}(\mathbb{R}^n).$$

Proof At first we remark that by (6.5) we can cover Γ with $\tilde{c}2^{jd}$ balls of radius 2^{-j}. That means Γ has a non-empty intersection with at most $c2^{jd}$ balls of radius 2^{-j} centered in $2^{-j}m$. Now we calculate the coefficients

$$k^{\beta}_{j,m}(\mu) = 2^{jn} \int_{\Gamma} (2^{j-J} y - 2^{-J} m)^{\beta} k(2^j y - m) \mu(y) dy.$$

If we put

$$q = \sup_{\substack{y\in\mathbb{R}^n \\ (2^j y - m)\in \operatorname{supp} k}} |2^{j-J}y - 2^{-J}m|,$$

then $q < 1$ holds, and it follows that

$$|k^\beta_{j,m}(\mu)| \leq c 2^{jn} q^{|\beta|} \mu(\Gamma \cap B_{2^{-j}m, 2^{J-j}}).$$

Now, we estimate

$$\|k(\mu)|l_p\|_{s_{\min}} \leq \left(\sum_j 2^{j(s(\Gamma)-n/p)p} \sum_{\beta,m} |k^\beta_{j,m}(\mu)|^p \right)^{1/p}$$

and, because of our remark above and $|q| < 1$, we obtain

$$\begin{aligned} \|k(\mu)|l_p\|_{s_{\min}} &\leq c \left(\sum_j 2^{j(-(n-d)/p'-\varepsilon-n/p)p} 2^{jd} 2^{jnp} 2^{-jdp} \right)^{1/p} \\ &\leq c \left(\sum_j 2^{-j\varepsilon p} \right)^{1/p} < \infty. \end{aligned}$$

Now we calculate the second term of the norm. In the case $\Gamma \cap B_{x,2^{-K+2}} = \emptyset$ we have by $j > J + K$, $K \geq J$ and $|2^{-j}m - \gamma| < 2^{J-j}$ for $\gamma \in \Gamma \cap B_{2^{-j}m,2^{J-j}}$

$$|x - 2^{-j}m| \geq 2^{-K+2} - 2^{-K} \geq 2^{-K+1} + 2^{-j}$$

and it follows that

$$B_{x,2^{-K+1}} \cap B_{2^{-j}m,2^{-j}} = \emptyset.$$

Therefore we can write

$$\sup_{K,x} 2^{-K(s_{K,x}-s_{\min})} \|k(\mu)|l_p\|^{K,x}_{s_{K,x}} = \sup_{\substack{K,x \\ \Gamma\cap B_{x,2^{-K+2}}\neq\emptyset}} 2^{-K(s_{K,x}-s_{\min})} \|k(\mu)|l_p\|^{K,x}_{s_{K,x}}.$$

But if $\Gamma \cap B_{x,2^{-K+2}} \neq \emptyset$, then the relation $s_{K,x} \leq s(\Gamma)$ holds, and with $2^{-K(s_{K,x}-s_{\min})} \leq 1$, we obtain

$$\sup_{K,x} 2^{-K(s_{K,x}-s_{\min})} \|k(\mu)|l_p\|^{K,x}_{s_{K,x}} \leq \left(\sum_{\beta,j,m} 2^{j(s(\Gamma)-n/p)p} |k^\beta_{j,m}(\mu)|^p \right)^{1/p} < \infty,$$

as already shown above.

□

For our next example we recall example (c) in subsection 4.1.2.
Let $g_\alpha(x) = \psi(x)|x|^{-\alpha}$ for $\psi \in C^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $\alpha \in \mathbb{R}$ with $n-1 < \alpha < n$. We put

$$f(x) = (D^\gamma g_\alpha)(x) \quad \text{for } \gamma \in \mathbb{N}^n_0 \text{ with } |\gamma| = 1.$$

Example 6.3 *Let $\mathbb{S} : x \mapsto s(x)$ be a lower semi-continuous function in $\mathbb{R}^n$ with*

$$s(x) = \begin{cases} n/p - \alpha - 1 - \varepsilon & : \quad x = 0 \\ -\varepsilon & : \quad x \neq 0 \end{cases}$$

for $\varepsilon > 0$ and $(n - \alpha - 1)p < -n$. Then

$$f \in B_p^{\mathbb{S}, s_{\min}}(\mathbb{R}^n).$$

Proof By the estimates we did in example (c), section 4.1.2, we have

$$\|k(f)|l_p\|_{s_{\min}} \leq \|k(f)|l_p\|_{s(0)} < \infty$$

if $(n - \alpha - 1)p < -n$. The interesting term is again the supremum. As a first case we treat all K, x with $0 \in B_{x,2^{-K+2}}$. Then $s_{K,x} \leq s(0)$ for all such K, x. Therefore,

$$\begin{aligned}
&\sup_{\substack{K,x \\ 0 \in B_{x,2^{-K+2}}}} 2^{-K(s_{K,x} - s_{\min})} \left(\sum_{\beta,j,m}^{K,x} 2^{j(s_{K,x} - n/p)p} |k_{j,m}^{\beta}(f)|^p \right)^{1/p} \\
&\quad \leq \sup_{\substack{K,x \\ 0 \in B_{x,2^{-K+2}}}} \left(\sum_{\beta,j,m}^{K,x} 2^{j(s(0) - n/p)p} |k_{j,m}^{\beta}(f)|^p \right)^{1/p} \\
&\quad \leq \left(\sum_{\beta,j,m} 2^{j(s(0) - n/p)p} |k_{j,m}^{\beta}(f)|^p \right)^{1/p} = \|k(f)|l_p\|_{s(0)} < \infty.
\end{aligned}$$

Now we treat the second case $0 \neq B_{x,2^{-K+2}}$. If we assume $|m| < 2^J$, then by $j > J + K$ and $K \geq J$ follows

$$|x - 2^{-j}m| \geq 2^{-K+2} - 2^{-K} \geq 2^{-K+1} + 2^{-j}$$

and we have

$$B_{x,2^{-K+1}} \cap B_{2^{-j}m,2^{-j}} = \emptyset.$$

Therefore, we can restrict ourselves to all $|m| \geq 2^J$, or more precise, by $j > J + K$ to all $|m| \geq 2^{j-K}$. On the other hand in example (c), subsection 4.1.2, we found, because of the integration conditions for

$$k_{j,m}^{\beta}(f) = \int\limits_{\substack{y \in \operatorname{supp} k \\ |2^{-j}m + 2^{-j}y| \leq 1}} (2^{-J}y)^{\beta} k(y) f(2^{-j}m + 2^{-j}y) dy,$$

that $|m| \leq 2^{j+1}$. Hence, in the second case we only need the coefficients $k_{j,m}^{\beta}(f)$ with $2^{j-K} \leq |m| \leq 2^{j+1}$. With $|y| < 2^J$ we find as a new integration condition, that $2^{-K-1} \leq |2^{-j}m + 2^{-j}y| \leq 1$. For $q = \sup_{y\in\operatorname{supp} k} |2^{-J}y|$ with $q < 1$ we estimate

$$\begin{aligned} |k^{\beta}_{j,m}(f)| &\leq cq^{|\beta|} \int\limits_{\substack{y\in\operatorname{supp} k \\ 2^{-K-1}\leq|2^{-j}m+2^{-j}y|\leq 1}} |D^{\gamma}(\psi(2^{-j}m + 2^{-j}y)|2^{-j}m + 2^{-j}y|^{-\alpha})|dy \\ &\leq cq^{|\beta|}2^{\alpha(K+1)}2^{Jn}. \end{aligned}$$

Now we can estimate the supremum in the second case

$$\sup_{\substack{K,x \\ 0\notin B_{x,2^{-K+2}}}} 2^{-K(s_{K,x}-s_{\min})} \Big(\sum_{\substack{\beta,j \\ 2^{j-K}\leq|m|\leq 2^{j+1}}}^{K,x} 2^{j(s_{K,x}-n/p)p}|k^{\beta}_{j,m}(f)|^p\Big)^{1/p}$$

$$\begin{aligned} &\leq c \sup_{\substack{K,x \\ 0\notin B_{x,2^{-K+2}}}} 2^{-K(s_{K,x}-s_{\min}-\alpha)} \left(\sum_{j=J+K+1}^{\infty} 2^{j(s_{K,x}-n/p)p}2^{jn}\right)^{1/p} \\ &\leq c \sup_{\substack{K,x \\ 0\notin B_{x,2^{-K+2}}}} 2^{-K(1-n/p)} \left(\sum_{j=J+1}^{\infty} 2^{-j\varepsilon p}\right)^{1/p} < \infty, \end{aligned}$$

because from $(n-\alpha-1)p < -n$ follows $p > n$. That proves the assertion.

□

Analogously one can prove a corresponding result for two or more separated singularities. Let $h(x) = g_{\alpha_1}(x) + g_{\alpha_2}(x - x_0)$ with $g(x)$ from Example 6.3, where now $n-1 < \alpha_1, \alpha_2 < n$ and $|x_0| > 8$. Then we put $f(x) = (D^{\gamma}h)(x)$ for $\gamma \in \mathbb{N}_0^n$ with $|\gamma| = 1$ and can state, that $f \in B_p^{\mathbb{S},s_{\min}}(\mathbb{R}^n)$, if

$$s(x) = \begin{cases} n/p - \alpha_1 - 1 - \varepsilon & : \quad x = 0 \\ n/p - \alpha_2 - 1 - \varepsilon & : \quad x = x_0 \\ -\varepsilon & : \quad \text{otherwise} \end{cases}$$

for $\varepsilon > 0$ and p sufficiently large.

6.3 Two-microlocal spaces

In this subsection we discuss the connection between the spaces of varying smoothness and the so-called two-microlocal spaces. These spaces were first defined by J.M. Bony in 1984, and have been studied in connection with wavelet methods by S. Jaffard and Y. Meyer, see [11] for details and references. We follow the approach given there. Let $\lambda^l_{j,m}(f)$ be the wavelet coefficients of $f \in S'(\mathbb{R}^n)$ in the decomposition (5.3), that we treated in Theorem 5.1.

Definition 6.1 *Let s and s' be two real numbers and $x^0 \in \mathbb{R}^n$. The two-microlocal space $C^{s,s'}(x^0)$ is the collection of all distributions f such that*

$$|\lambda^l_{j,m}(f)| \le c2^{-js}(1 + |m - 2^j x^0|)^{-s'}, \tag{6.6}$$

for all $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $1 \le l \le L$.

This definition of $C^{s,s'}(x^0)$ is given in [11], Proposition 1.4., as an equivalent characterization. The original one was formulated before in terms of the Littlewood-Paley decomposition, where also a discussion about basic properties can be found. We will treat the case $s' \ge 0$, which corresponds to our notion of lower semi-continuous functions to describe a situation where a distribution has a singularity at the point x^0 and is smoother in a neighbourhood. Now we state the Theorem which shows the connection of the two-microlocal spaces and the spaces of varying smoothness.

Theorem 6.4 *Let $s' \ge 0$ and $f \in C^{s,s'}(x^0)$. Then*

$$f \in B^{\mathbb{S},s_0}_\infty(\mathbb{R}^n) \quad \text{with } s_0 < 0 \text{ and } s(x) \le \begin{cases} s & : \ x = x^0 \\ s + s' & : \ \text{otherwise.} \end{cases}$$

Here the meaning of the two parameters s and s' becomes clear. The smoothness in the point x^0 is described by s and its difference to the smoothness in a neighbourhood around x^0 is described by s'.

Proof We use Theorem 6.3. The first term of the right-hand side of (6.4) is easy to estimate,

$$\|\lambda(f)|b^{s_{\min}}_\infty\| = \sup_{l,j,m} 2^{js_{\min}}|\lambda^l_{j,m}(f)| \le \sup_{l,j,m} 2^{js}c2^{-js} < \infty.$$

The norm in the supremum term on the right-hand side of (6.4) reads as

$$\|\lambda(f)|b^{s_{K,x}}_\infty\|^{K,x} = \sup_{l,j,m}{}^{K,x} 2^{js_{K,x}}|\lambda^l_{j,m}(f)|,$$

where the supremum is taken over all l, m and $j > \tilde{J} + K$ with

$$B_{x,2^{-K+1}} \cap B_{2^{-j}m,2^{-j}} \neq \emptyset. \tag{6.7}$$

To estimate this norm we distinguish two cases. As the first case we treat all x, K with $|x - x^0| \le 2^{-K+2}$. Then we know that $s_{K,x} \le s(x^0) \le s$. Therefore we have

$$\|\lambda(f)|b^{s_{K,x}}_\infty\|^{K,x} \le \sup_{l,j,m}{}^{K,x} 2^{js}c2^{-js} \le c.$$

In the other case we have $|x - x^0| > 2^{-K+2}$ and because of (6.7) we can estimate

$$|m - 2^j x^0| \ge |2^j(x - x^0)| - |2^j x - m| \ge 2^{j-K+2} - 2^{j-K+1} - 1 \ge 2^{j-K+1} - 1.$$

Therefore we get

$$\begin{aligned}\|\lambda(f)|b_\infty^{s_{K,x}}\|^{K,x} &\leq \sup_{l,j,m}{}^{K,x} 2^{js_{K,x}} c 2^{-js}(1+|m-2^j x_0|)^{-s'} \\ &\leq c \sup_{j>\tilde{J}+K} 2^{(j-K)s_{K,x}} 2^{-(j-K)s} 2^{-(j-K)s'} 2^{K(s_{K,x}-s)} \\ &\leq c 2^{K(s_{K,x}-s)},\end{aligned}$$

where we used $s_{K,x} \leq s+s'$ in the last line. Finally we obtain for the supremum on the right-hand side of (6.4)

$$\sup_{x,K\geq\tilde{J}} 2^{-K(s_{K,x}-s_{\min})}\|\lambda(f)|b_\infty^{s_{K,x}}\|^{K,x} \leq c \sup_{x,K\geq\tilde{J}} 2^{-K(s-s_{\min})} < \infty,$$

which proves the assertion.

□

It would be desirable to have also the other direction, this would mean that the two involved spaces are equal. But that can not be expected, because for coefficients $\lambda^l_{j,m}(f)$, where $2^{-j}m$ is far away from x^0, condition (6.6) is to strong to hold for a function $f \in B_\infty^{\mathbb{S},s_{\min}}(\mathbb{R}^n)$. Nevertheless we can prove the other direction in terms of a local version of $C^{s,s'}(x^0)$. We say that a function f belongs to $C^{s,s'}_{loc}(x^0)$ if there exists a neighborhood U_{x^0} of x^0 and a function $h \in C^{s,s'}(x^0)$ such that $f = h$ on U_{x^0}, see also [11](p.15).

Theorem 6.5 *Let $s<0$, $s'\geq 0$ and $f \in B_\infty^{\mathbb{S},s}(\mathbb{R}^n)$ with*

$$s(x) = \begin{cases} s & : \quad x = x^0 \\ s+s' & : \quad \textit{otherwise.}\end{cases}$$

Then $f \in C^{s,s'}_{loc}(x^0)$.

Proof We start with some short preparations. It is sufficient to prove that $\varphi_{x^0} f \in C^{s,s'}(x^0)$ for a C^∞-function $\varphi_{x^0}(x)$ with

$$\varphi_{x^0}(x) = 1 \quad \text{for } |x-x^0| \leq 1 \quad \text{and} \quad \varphi_{x^0}(x) = 0 \quad \text{for } |x-x^0| > 2.$$

By Theorem 3.3 we know if $f \in B_\infty^{\mathbb{S},s}(\mathbb{R}^n)$ for $s<0$ then we also have $\varphi_{x^0} f \in B_\infty^{\mathbb{S},s}(\mathbb{R}^n)$, hence, it is even enough to show that $g \in C^{s,s'}(x^0)$ for any compactly supported function $g \in B_\infty^{\mathbb{S},s}(\mathbb{R}^n)$. We assume $\operatorname{supp} g \subset B_{x^0,1}$, then we have

$$|\lambda^l_{j,m}(g)| = 0 \quad \text{if} \quad |2^{-j}m - x^0| \geq 1 + 2^{\tilde{J}-j}.$$

So we only care about the coefficients for which $|2^{-j}m - x^0| \leq 1 + 2^{\tilde{J}-j}$. If, in addition, $j < j_0$ for a $j_0 \in \mathbb{N}$ we get by Theorem 6.3

$$|\lambda^l_{j,m}(g)| \leq c2^{-js} \leq c2^{-js}(1+|2^j x^0 - m|)^{-s'},$$

because $s' \geq 0$. We choose $j_0 = \tilde{J} + 4$ and divide the estimate for the coefficients in the case $j \geq j_0$ into two steps.

Step 1

As a first case we treat all coefficients $\lambda^l_{j,m}(g)$ such that we can find a number $i \in \{\tilde{J}+4, \tilde{J}+5, \ldots, j\}$ with

$$2^{-j+i} < |2^{-j}m - x^0| \leq 2^{-j+i+1}.$$

Then we have for $x_1 = 2^{-j}m$ and $K_1 = j - i + 3$

$$B_{x_1, 2^{-K_1-1}} \cap B_{2^{-j}m, 2^{-j}} \neq \emptyset \quad \text{and} \quad |x_1 - x^0| > 2^{-K_1+2}.$$

Now Theorem 6.3 gives

$$|\lambda^l_{j,m}(g)| \leq c 2^{K_1(s_{K_1,x_1} - s)} 2^{-j s_{K_1,x_1}} = c 2^{(j-i+3)(s+s'-s)} 2^{-j(s+s')} \leq c 2^{-js} 2^{-is'}, \quad (6.8)$$

because $s_{K_1,x_1} = s + s'$ holds in that case. Furthermore, we know

$$1 + |2^j x^0 - m| \leq 1 + 2^{i+1} \leq c 2^i.$$

Inserting that into (6.8) we obtain

$$|\lambda^l_{j,m}(g)| \leq c 2^{-js} (1 + |2^j x^0 - m|)^{-s'}.$$

Step 2

For all the remaining coefficients we have

$$|2^{-j}m - x^0| \leq 2^{-j+\tilde{J}+4}.$$

Then for $x_2 = 2^{-j}m$ and $K_2 = j - \tilde{J} - 2$

$$B_{x_2, 2^{-K_2-1}} \cap B_{2^{-j}m, 2^{-j}} \neq \emptyset \quad \text{and} \quad |x_2 - x^0| < 2^{-K_2+2}$$

hold. Theorem 6.3 gives

$$|\lambda^l_{j,m}(g)| \leq c 2^{K_2(s_{K_2,x_2} - s)} 2^{-j s_{K_2,x_2}} = c 2^{-js}, \quad (6.9)$$

because now $s_{K_2,x_2} = s$ holds. We can also estimate

$$1 + |2^j x^0 - m| \leq 1 + 2^{\tilde{J}+4} \leq c.$$

Therefore we obtain

$$|\lambda^l_{j,m}(g)| \leq c 2^{-js} (1 + |2^j x^0 - m|)^{-s'}$$

also in this case. That completes the proof.

□

So far we only treated the case $p = \infty$. A more general definition for two-microlocal spaces was given by Moritoh and Yamada in [18], where they treated homogeneous spaces $B^{s,s'}_{p,q}(U)$ for $1 \leq p, q \leq \infty$, $s > 0, s' \in \mathbb{R}$ and an open subset $U \subset \mathbb{R}^n$. We give now a modified version of this definition.

Definition 6.2 *Let s and s' be two real numbers, $1 < p < \infty$ and $x^0 \in \mathbb{R}^n$. The two-microlocal space $B_p^{s,s'}(x^0)$ is the collection of all distributions $f \in S'(\mathbb{R}^n)$ such that*

$$\|f|B_p^{s,s'}(x^0)\| = \left(\sum_{j\in\mathbb{N}_0} 2^{j(s-n/p)p} \sum_{m\in\mathbb{Z}^n} (1+|2^j x^0 - m|)^{s'p} |\lambda_{j,m}^l(f)|^p \right)^{1/p} < \infty.$$

Again we restrict ourselves to $s' \geq 0$ in order to prove the following connection to the spaces of varying smoothness, where the smoothness behavior is characterized by a lower semi-continuous function.

Theorem 6.6 *Let $s' \geq 0$ and $f \in B_p^{s,s'}(x^0)$. Then*

$$f \in B_p^{\mathbb{S},s_0}(\mathbb{R}^n) \quad \text{with } s_0 < 0 \text{ and } s(x) \leq \begin{cases} s & : \; x = x^0 \\ s+s' & : \; \text{otherwise.} \end{cases}$$

Proof We use Theorem 6.3 again. Because $s_{\min} \leq s$ and $s' \geq 0$ we get immediately

$$\|\lambda(f)|b_p^{s_{\min}}\| = \left(\sum_{j,m,l} 2^{j(s_{\min}-n/p)p} |\lambda_{j,m}^l(f)|^p \right)^{1/p} \leq \|f|B_p^{s,s'}(x^0)\| < \infty.$$

In order to estimate the supremum term on the right-hand side of (6.4) we discuss two cases. As a first case we treat all x and $K \geq \tilde{J}$ such that $|x - x^0| \leq 2^{-K+2}$. Then we know $s_{K,x} \leq s$ and can estimate

$$\|\lambda(f)|b_p^{s_{K,x}}\|^{K,x} \leq \left(\sum_{j,m,l}^{K,x} 2^{j(s-n/p)p} |\lambda_{j,m}^l(f)|^p \right)^{1/p} \leq \|f|B_p^{s,s'}(x^0)\| \leq c.$$

In the second case we treat all x and $K \geq \tilde{J}$ with $|x-x^0| > 2^{-K+2}$. Then we know $s_{K,x} \leq s + s'$. Furthermore, under the assumption $|x - 2^{-j}m| \leq 2^{-K+1} + 2^{-j}$ for $j > \tilde{J} + K$ we have $1 + |2^j x^0 - m| \geq 2^{j-K}$. Now we can estimate in the following way

$$\begin{aligned} &\|\lambda(f)|b_p^{s_{K,x}}\|^{K,x} \\ &= \left(\sum_{j,m,l}^{K,x} 2^{j(s_{K,x}-n/p)p} (1+|2^j x^0 - m|)^{s'p} (1+|2^j x_0 - m|)^{-s'p} |\lambda_{j,m}^l(f)|^p \right)^{1/p} \\ &\leq \left(\sum_{j,m,l}^{K,x} 2^{(j-K)(s_{K,x}-n/p)p} 2^{K(s_{K,x}-n/p)p} (1+|2^j x^0 - m|)^{s'p} 2^{-(j-K)s'p} |\lambda_{j,m}^l(f)|^p \right)^{1/p} \end{aligned}$$

$$\leq \quad 2^{K(s_{K,x}-n/p)} \left(\sum_{j,m,l}^{K,x} 2^{(j-K)(s_{K,x}-s'-n/p)p} (1+|2^j x^0 - m|)^{s'p} |\lambda^l_{j,m}(f)|^p \right)^{1/p}$$

$$\leq \quad 2^{K(s_{K,x}-n/p)} 2^{-K(s-n/p)} \left(\sum_{j,m,l}^{K,x} 2^{j(s-n/p)p} (1+|2^j x^0 - m|)^{s'p} |\lambda^l_{j,m}(f)|^p \right)^{1/p}$$

$$\leq \quad 2^{K(s_{K,x}-s)} \|f|B^{s,s'}_p(x^0)\| \leq c 2^{K(s_{K,x}-s)},$$

because $s_{K,x} - s' \leq s$. Therefore we obtain for the supremum term on the right-hand side of (6.4)

$$\sup_{x,K \geq \tilde{J}} 2^{-K(s_{K,x}-s_{\min})} \|\lambda(f)|b^{s_{K,x}}_p\|^{K,x} \leq c \sup_{K \geq \tilde{J}} 2^{-K(s-s_{\min})} < \infty,$$

which proves the desired assertion.

□

As in the case $p = \infty$, by the same arguments as there, it is not possible to get the converse result. Therefore we define again the local version $B^{s,s'}_{p,loc}(x^0)$ by the same restriction procedure as before. But not even the corresponding weaker result, in analogy to Theorem 6.5, can be expected. We briefly explain the reason. In Step 1 of the proof of Theorem 6.5 we had to distinguish between the coefficients with

$$2^{-j+i} < |2^{-j} m - x^0| \leq 2^{-j+i+1}$$

for different numbers i to use information from Theorem 6.3. We would have to go the same way now, but then we would get an additional sum over all these numbers i, such that the left-hand side of (6.4) would not dominate the norm in $B^{s,s'}_{p,loc}(x^0)$ up to a constant. It turns out, that this additional sum does not matter if we slightly decrease the smoothness in the target space. We can state the following. Let $s < 0$, $s' \geq 0$ and $f \in B^{\mathbb{S},s}_p(\mathbb{R}^n)$ with

$$s(x) = \begin{cases} s & : \quad x = x^0 \\ s + s' & : \quad \text{otherwise.} \end{cases}$$

Then $f \in B^{\tilde{s},s'}_{p,loc}(x^0)$ with $\tilde{s} < s$.

7 Further problems

The investigations of this section are essentially based on the results proven in the last section. The norm estimates given there will be extremely helpful for the problems we are going to treat now.

7.1 Sharp embeddings

A standard challenge when dealing with function spaces is to give sharp embedding conditions. Now we will prove that the conditions appearing in Theorem 3.4 and Corollary 3.1 are not only sufficient but also necessary, at least if $s_0 = s_{\min} < 0$. We start with the simpler case $p_1 = p_2 = p$.

Theorem 7.1 *Let $1 < p \leq \infty$ and let $\mathbb{S}^1$ and $\mathbb{S}^2$ be bounded lower semi-continuous functions in $\mathbb{R}^n$ with $s^1_{\min}, s^2_{\min} < 0$.*

$$\text{If} \quad B_p^{\mathbb{S}^1, s^1_{\min}}(\mathbb{R}^n) \subset B_p^{\mathbb{S}^2, s^2_{\min}}(\mathbb{R}^n), \quad \text{then} \quad s^1(x) \geq s^2(x) \text{ for all } x \in \mathbb{R}^n.$$

Proof We assume that there exists a point $x^0 \in \mathbb{R}^n$, such that $s^1(x^0) < s^2(x^0)$ holds. Then there is a neighborhood U_{x^0} of x^0 with the properties

$$\begin{aligned} \inf_{x \in U_{x^0}} s^1(x) &\leq s^1(x^0) \\ \inf_{x \in U_{x^0}} s^2(x) &\geq s^1(x^0) + \delta \quad \text{for } \delta > 0, \end{aligned}$$

(see formula (3.1)). Let us assume $x^0 = 0$, $U_0 = B_r$ with $r > 0$, then we define for $0 < \varepsilon < \delta$ the coefficients

$$\lambda^0_{j,0}(f) = \begin{cases} 2^{j(n/p - s^1(0) - \varepsilon)} & : \quad B_{2^{-j}} \subset U_0 \\ 0 & : \quad \text{otherwise} \end{cases}$$

and $\lambda^l_{j,m}(f) = 0$ for $l, m \neq 0$. In the case $0 \notin B_{x,2^{-K+2}}$, then by $j > \tilde{J} + K$ we have

$$B_{x,2^{-K+1}} \cap B_{2^{-j}} = \emptyset.$$

Therefore we can write

$$\sup_{K,x} 2^{-K(s^1_{K,x} - s^1_{\min})} \|\lambda(f)|b_p^{s^1_{K,x}}\|^{K,x} = \sup_{\substack{K,x \\ 0 \in B_{x,2^{-K+2}}}} 2^{-K(s^1_{K,x} - s^1_{\min})} \|\lambda(f)|b_p^{s^1_{K,x}}\|^{K,x}.$$

We know that for all K and x with $0 \in B_{x,2^{-K+2}}$, the relation $s^1_{K,x} \le s^1(0)$ holds, hence, for $K \ge \tilde{J}$, we get

$$\sup_{\substack{K,x \\ 0\in B_{x,2^{-K+2}}}} 2^{-K(s^1_{K,x}-s^1_{\min})} \left(\sum_{j>\tilde{J}+K}^{K,x} 2^{j(s^1_{K,x}-n/p)p} |\lambda^0_{j,0}(f)|^p \right)^{1/p}$$

$$\le \left(\sum_{j=2\tilde{J}+1}^{\infty} 2^{j(s^1(0)-n/p)p} 2^{j(n/p-s^1(0)-\varepsilon)p} \right)^{1/p}$$

$$= \left(\sum_{j=2\tilde{J}+1}^{\infty} 2^{-j\varepsilon p} \right)^{1/p} < \infty,$$

where, again we used $2^{-K(s^1_{K,x}-s^1_{\min})} \le 1$. Furthermore

$$\|\lambda(f)|b_p^{s^1_{\min}}\| \le \|\lambda(f)|b_p^{s^1(0)}\| < \infty.$$

On the other hand, we calculate

$$\sup_{K,x} 2^{-K(s^2_{K,x}-s^2_{\min})} \left(\sum_{j>\tilde{J}+K+2}^{K+2,x} 2^{j(s^2_{K,x}-n/p)p} |\lambda^0_{j,0}(f)|^p \right)^{1/p}$$

$$\ge 2^{-\tilde{K}(s^2_{\tilde{K},0}-s^2_{\min})} \left(\sum_{j=\tilde{J}+\tilde{K}+1}^{\infty} 2^{j(s^1(0)+\delta-n/p)p} 2^{j(n/p-s^1(0)-\varepsilon)p} \right)^{1/p}$$

$$= 2^{-\tilde{K}(s^2_{\tilde{K},0}-s^2_{\min})} \left(\sum_{j=\tilde{J}+\tilde{K}+1}^{\infty} 2^{j(\delta-\varepsilon)p} \right)^{1/p} = \infty,$$

where we chose $\tilde{K}$ sufficiently large. But now it follows with both sides of (6.4) that

$$f \in B_p^{\mathbb{S}^1, s^1_{\min}}(\mathbb{R}^n) \quad \text{but} \quad f \notin B_p^{\mathbb{S}^2, s^2_{\min}}(\mathbb{R}^n),$$

which proves the assertion.

□

Theorem 7.2 *Let $1 < p_1 < p_2 \le \infty$ and let $\mathbb{S}^1$ and $\mathbb{S}^2$ be bounded lower semi-continuous functions in $\mathbb{R}^n$ with $s^1_{\min}, s^2_{\min} < 0$.*

If $B_{p_1}^{\mathbb{S}^1, s^1_{\min}}(\mathbb{R}^n) \subset B_{p_2}^{\mathbb{S}^2, s^2_{\min}}(\mathbb{R}^n)$ *then* $s^1(x) - \dfrac{n}{p_1} \ge s^2(x) - \dfrac{n}{p_2}$ *for all* $x \in \mathbb{R}^n$.

Proof The idea of this proof is the same as for the proof of Theorem 7.1. We put

$$\tilde{s}^1(x) = s^1(x) - n/p_1 \quad \text{and} \quad \tilde{s}^2(x) = s^2(x) - n/p_2$$

and assume that there exists a point $x^0 \in \mathbb{R}^n$, such that $\tilde{s}^1(x^0) < \tilde{s}^2(x^0)$ holds. Then there is a neighborhood U_{x^0} of x^0 with the properties

$$\begin{aligned} \inf_{x \in U_{x^0}} \tilde{s}^1(x) &\leq \tilde{s}^1(x^0) \\ \inf_{x \in U_{x^0}} \tilde{s}^2(x) &\geq \tilde{s}^1(x^0) + \delta \quad \text{for } \delta > 0, \end{aligned}$$

(see formula (3.1)). Let us assume $x^0 = 0$, $U_0 = B_r$ with $r > 0$, then we define for $0 < \varepsilon < \delta$ the coefficients

$$\lambda^0_{j,0}(f) = \begin{cases} 2^{-j(\tilde{s}^1(0)+\varepsilon)} & : \quad B_{2^{-j}} \subset U_0 \\ 0 & : \quad \text{otherwise} \end{cases}$$

and $\lambda^l_{j,m}(f) = 0$ for $l, m \neq 0$. In the case $0 \notin B_{x,2^{-K+2}}$, then by $j > \tilde{J} + K$ we have

$$B_{x,2^{-K+1}} \cap B_{2^{-j}} = \emptyset.$$

Therefore we can write

$$\sup_{K,x} 2^{-K(s^1_{K,x} - s^1_{\min})} \|\lambda(f)|b^{s^1_{K,x}}_{p_1}\|^{K,x} = \sup_{\substack{K,x \\ 0 \in B_{x,2^{-K+2}}}} 2^{-K(s^1_{K,x} - s^1_{\min})} \|\lambda(f)|b^{s^1_{K,x}}_{p_1}\|^{K,x}.$$

We know that for all K and x with $0 \in B_{x,2^{-K+2}}$, the relation $\tilde{s}^1_{K,x} \leq \tilde{s}^1(0)$ holds, hence, for $K \geq \tilde{J}$, we get

$$\begin{aligned} \sup_{\substack{K,x \\ 0 \in B_{x,2^{-K+2}}}} \quad 2^{-K(s^1_{K,x} - s^1_{\min})} &\left(\sum_{j > \tilde{J}+K}^{K,x} 2^{j\tilde{s}^1_{K,x} p_1} |\lambda^0_{j,0}(f)|^{p_1} \right)^{1/p_1} \\ &\leq \left(\sum_{j=2\tilde{J}+1}^{\infty} 2^{j\tilde{s}^1(0)p_1} 2^{-j(\tilde{s}^1(0)+\varepsilon)p_1} \right)^{1/p_1} \\ &= \left(\sum_{j=2\tilde{J}+1}^{\infty} 2^{-j\varepsilon p_1} \right)^{1/p_1} < \infty, \end{aligned}$$

where, again we used $2^{-K(s^1_{K,x} - s^1_{\min})} \leq 1$. Furthermore

$$\|\lambda(f)|b^{s^1_{\min}}_{p_1}\| \leq \|\lambda(f)|b^{s^1(0)}_{p_1}\| < \infty.$$

On the other hand, we estimate

$$\sup_{K,x} 2^{-K(s^2_{K,x}-s^2_{\min})} \left(\sum_{j>\tilde{J}+K+2}^{K+2,x} 2^{j\tilde{s}^2_{K,x}p_2} |\lambda^0_{j,0}(f)|^{p_2} \right)^{1/p_2}$$

$$\geq \quad 2^{-\tilde{K}(s^2_{\tilde{K},0}-s^2_{\min})} \left(\sum_{j=\tilde{J}+\tilde{K}+1}^{\infty} 2^{j(\tilde{s}^1(0)+\delta)p_2} 2^{-j(\tilde{s}^1(0)+\varepsilon)p_2} \right)^{1/p_2}$$

$$= \quad 2^{-\tilde{K}(s^2_{\tilde{K},0}-s^2_{\min})} \left(\sum_{j=\tilde{J}+\tilde{K}+1}^{\infty} 2^{j(\delta-\varepsilon)p_2} \right)^{1/p_2} = \infty,$$

where we chose $\tilde{K}$ sufficiently large. But now it follows with both sides of (6.4) that

$$f \in B^{\mathbb{S}^1, s^1_{\min}}_{p_1}(\mathbb{R}^n) \quad \text{but} \quad f \notin B^{\mathbb{S}^2, s^2_{\min}}_{p_2}(\mathbb{R}^n),$$

which proves the assertion.

□

7.2 A special construction

In this section we try to answer the following question. Given a lower semi-continuous function $\mathbb{S} : x \mapsto s(x)$ in $\mathbb{R}^n$, is it possible to construct a function f with the properties

$$f \in B^{\mathbb{S}, s_{\min}}_{p}(\mathbb{R}^n) \quad \text{and} \quad f \notin B^{\mathbb{S}+\varrho, s_{\min}}_{p}(\mathbb{R}^n)$$

for every non-negative lower semi-continuous function $\varrho = \varrho(x) \neq 0$?
We try to give a partial answer to this question. We generalize the construction of Theorem 16.2 in [24] for the one-dimensional case. Let

$$\omega(x) = e^{-\frac{1}{1-4x^2}} \text{ if } |x| < 1/2 \quad \text{and} \quad \omega(x) = 0 \text{ otherwise}, \quad x \in \mathbb{R},$$

be the $C^\infty_0(\mathbb{R})$ standard function. Let $0 < s < 1$ and $\nu_i = 2^{\kappa i}$ for $i \in \mathbb{N}$ and $\kappa \in \mathbb{R}_+$ with $2^\kappa s > 1$. Then we set

$$f = \sum_{k=0}^{\infty} \sum_{i=1}^{\infty} f_{k,i}, \quad \text{with} \quad f_{k,i}(x) = 2^{-\nu_i s} \sum_{l}^{i} \omega(2^{\nu_i} x - l), \tag{7.1}$$

where we sum over those $l \in \mathbb{Z}$, such that $Q_{\nu_i,l} \subset Q_k$. Here we used the notation $Q_{\nu,l}$ for the interval $[2^{-\nu}l - 2^{-\nu}, 2^{-\nu}l + 2^{-\nu}]$ and $Q_k = [2^{-2k-1}, 2^{-2k})$. The construction gives

$$\operatorname{supp} f \subset [0,1] \quad \text{with} \quad \operatorname{supp} f_{k,i} \subset Q_k \quad \text{for all} \quad i \in \mathbb{N}.$$

We know by the Theorem 13.8 in [24], that the function f belongs to $B^s_\infty(\mathbb{R})$. At first we shall check that its first derivative f' belongs to $B^{s-1}_\infty(\mathbb{R})$ in terms of our characterisation (4.11). Therefore, we have to estimate the coefficients $k^\beta_{j,m}(f')$, where we use the abbreviation $I_{j,m}$ for the interval $[2^{-j}m - 2^{J-j}, 2^{-j}m + 2^{J-j}]$.
Case 1: $\mathrm{dist}(0, I_{j,m}) < 2^{-j}$
Here we integrate in the distributional sense

$$k^\beta_{j,m}(f') = 2^j \int\limits_{I_{j,m}} (2^{j-J}y - 2^{-J}m)^\beta k(2^j y - m) f'(y) dy.$$

After a simple calculation we get by $|2^j y - m| < 2^J$

$$|k^\beta_{j,m}(f')| \leq c 2^{2j} \int\limits_{I_{j,m}} f(y) dy.$$

Now we treat the remaining integral

$$\int\limits_{I_{j,m}} \sum_{k=0}^\infty \sum_{i=1}^\infty 2^{-\nu_i s} \sum_l^i \omega(2^{\nu_i} y - l) dy. \tag{7.2}$$

Since $\mathrm{dist}(0, I_{j,m}) < 2^{-j}$, we need for the integration at most those Q_k with $k \gtrsim (j-J)/2$. Furthermore, because of $Q_{\nu_i,l} \subset Q_k$, we have the condition $\nu_i \gtrsim 2k$. That gives

$$(7.2) \quad \leq \sum_{k \gtrsim \frac{j-J}{2}} \sum_{\substack{i \in \mathbb{N} \\ \nu_i \gtrsim 2k}} 2^{-\nu_i s} \int\limits_{Q_k} \sum_l^i \omega(2^{\nu_i} y - l) dy.$$

Now, for fixed i, k, we ask, how many $l \in \mathbb{Z}$ there are with $Q_{\nu_i,l} \subset Q_k$, and we count $\sim 2^{-2k+\nu_i}$. Moreover, we estimate

$$\int\limits_{Q_{\nu_i,l}} \omega(2^{\nu_i} y - l) dy \leq c 2^{-\nu_i}.$$

Altogether we have

$$(7.2) \quad \leq c \sum_{k \gtrsim \frac{j-J}{2}} \sum_{\substack{i \in \mathbb{N} \\ \nu_i \gtrsim 2k}} 2^{-\nu_i s} 2^{-2k+\nu_i} 2^{-\nu_i} \leq c 2^{-j-js}$$

and get the following estimate

$$|k^\beta_{j,m}(f')| \leq c 2^{2j} 2^{-j-js} \leq c 2^{-j(s-1)}. \tag{7.3}$$

Case 2: $\operatorname{dist}(0, I_{j,m}) \geq 2^{-j}$
We write

$$f = f_1 + f_2, \quad \text{with} \quad f_1 = \sum_{k=0}^{\infty} \sum_{i=1}^{[I]} f_{k,i} \quad \text{and} \quad f_2 = \sum_{k=0}^{\infty} \sum_{i=[I]+1}^{\infty} f_{k,i}, \tag{7.4}$$

where $[I]$ is the whole part of a non-negative real number I which we choose later on. Because of the linearity of the coefficients $k_{j,m}^{\beta}$, we have

$$k_{j,m}^{\beta}(f') = k_{j,m}^{\beta}(f_1') + k_{j,m}^{\beta}(f_2').$$

We start to estimate the coefficients for f_1',

$$k_{j,m}^{\beta}(f_1') = \int_{\operatorname{supp} k} (2^{-J}y)^{\beta} k(y) f_1'(2^{-j}m + 2^{-j}y)dy.$$

Since $\operatorname{dist}(0, I_{j,m}) \geq 2^{-j}$, we only need a finite number of Q_k's for the integration, therefore, we integrate in the usual sense and get

$$|k_{j,m}^{\beta}(f_1')| \leq c \sum_{k=0}^{j/2} \sum_{i=1}^{[I]} 2^{-\nu_i(s-1)} \int_{2^j Q_k - m} \sum_{l}^{i} |\omega'(2^{\nu_i}(2^{-j}m + 2^{-j}y) - l)| dy. \tag{7.5}$$

Now we use the same arguments as above and can estimate

$$\begin{aligned} |k_{j,m}^{\beta}(f_1')| &\leq c \sum_{k=0}^{j/2} \sum_{i=1}^{[I]} 2^{-\nu_i(s-1)} 2^{-2k+\nu_i} 2^{-\nu_i} \\ &\leq c 2^{-\nu_{[I]}(s-1)} \leq c 2^{-2^{\kappa I}(s-1)} \leq c 2^{-j(s-1)}, \end{aligned} \tag{7.6}$$

where we chose $2^{\kappa I} = j$ to arrive at the desired estimate.
For the coefficients of f_2' we integrate in the distributional sense and get

$$|k_{j,m}^{\beta}(f_2')| \leq c 2^{2j} \int_{I_{j,m}} \left| \sum_{k=0}^{\infty} \sum_{i=[I]+1}^{\infty} f_{k_i}(y) \right| dy. \tag{7.7}$$

The construction of the f_{k_i} gives

$$\sup_y \left| \sum_{i=[I]+1}^{\infty} f_{k_i}(y) \right| \leq c 2^{-\nu_{[I]+1} s}$$

and because the functions f_{k_i} have disjoint supports for different k we find

$$\begin{aligned}
|k^\beta_{j,m}(f_2')| &\le c2^{2j}|I_{j,m}| \sup_{y\in I_{j,m}} \left(\sum_{k=0}^{\infty} \left| \sum_{i=[I]+1}^{\infty} f_{k_i}(y) \right| \right) \\
&\le c2^{2j}2^{-j}2^{-\nu_{[I]+1}s} \\
&\le c2^{j}2^{-js} \\
&\le c2^{-j(s-1)}, \qquad (7.8)
\end{aligned}$$

where we used $2^{\kappa I} = j$ again.
Finally, (7.3) together with (7.6) and (7.8) prove $f' \in B^{s-1}_\infty(\mathbb{R})$.
Now we generalize the construction (7.1) and define for a monotone increasing sequence $(s_k)_{k=0}^\infty$, with $0 < s_k < 1$ for all k , the function

$$F = \sum_{k=0}^{\infty}\sum_{i=1}^{\infty} F_{k,i}, \qquad \text{with} \qquad F_{k,i} = 2^{-\nu_i s_k}\sum_{l}^{i} \omega(2^{\nu_i}x - l),$$

with the same restrictions for the sum over l as in (7.1). By the previous calculations we are able to prove the following.

Example 7.1 *Let $\mathbb{S} : x \mapsto s(x)$ be a lower semi-continuous function in $\mathbb{R}$ with*

$$s(x) = \begin{cases} s_k - 1 & : \quad x \in Q_k \\ -\varepsilon & : \quad \textit{otherwise} \end{cases}$$

for $\varepsilon > 0$. Then $F' \in B^{\mathbb{S},s_{\min}}_\infty(\mathbb{R})$.

Here F' denotes the first derivative of F.
Proof
We have to check, that

$$\|k(F')|l_\infty\|_{s_{\min}} + \sup_{x,K\ge J} 2^{-K(s_{K,x}-s_{\min})}\|k(F')|l_\infty\|^{K,x}_{s_{K,x}} < \infty.$$

We start with the first part

$$\|k(F')|l_\infty\|_{s_{\min}} = \sup_{\beta,j,m} 2^{js_{\min}}|k^\beta_{j,m}(F')|.$$

If we substitute s by s_k in (7.2) and (7.5), increase the factor $2^{-\nu_i s_k}$ (or $2^{-\nu_i(s_k-1)}$) by $2^{-\nu_i s_0}$ (or $2^{-\nu_i(s_0-1)}$) and follow the previous calculation, then we have by (7.3), (7.6) and (7.8)

$$|k^\beta_{j,m}(F')| \le c2^{-j(s_0-1)}. \qquad (7.9)$$

Because of $s_{\min} \leq s_0 - 1$ it follows, that $\|k(F')|l_\infty\|_{s_{\min}} < \infty$.
Now we take care about the second part and check, if

$$\sup_{x,K\geq J} 2^{-K(s_{K,x}-s_{\min})} \sup_{\beta,j,m}{}^{K,x} 2^{js_{K,x}}|k^{\beta}_{j,m}(F')| < \infty, \tag{7.10}$$

where the supremum is taken over all

$$m, j > J + K \quad \text{with} \quad B_{x,2^{-K+1}} \cap B_{2^{-j}m,2^{-j}} \neq \emptyset.$$

We split the supremum over x, K in (7.10) into the five cases A-E.
A: $B_{x,2^{-K+2}} \cap [0, 1] = \emptyset$.
That is the trivial case, because all the coefficients $k^{\beta}_{j,m}(F')$ are zero.
B: $B_{x,2^{-K+2}} \supset [0, 1]$.
In this case we have $s_{K,x} \leq s_0 - 1$ and (7.9) answers the question in (7.10).
C: $B_{x,2^{-K+2}}$ has a non-empty intersection with at most one Q_k.
Then we know $s_{K,x} = s_k - 1$. We use the idea of Case 2 and split F into F_1 and F_2 as in (7.4). Then we get similar to (7.5) and (7.6)

$$\begin{aligned} |k^{\beta}_{j,m}(F'_1)| &\leq c\sum_{i=1}^{[I]} 2^{-\nu_i(s_k-1)} \int\limits_{Q_k} \sum_{l}^{i} |\omega'(2^{\nu_i}(2^{-j}m + 2^{-j}y) - l)|dy \\ &\leq c\sum_{i=1}^{[I]} 2^{-\nu_i(s_k-1)} 2^{-2k+\nu_i} 2^{-\nu_i} \\ &\leq c2^{-\nu_{[I]}(s_k-1)} \leq c2^{-2^{\kappa I}(s_k-1)} \leq c2^{-j(s_k-1)}, \end{aligned}$$

which is the desired estimate, if we choose $2^{\kappa I} = j$ again. Furthermore, as in (7.7) and (7.8), we have

$$\begin{aligned} |k^{\beta}_{j,m}(F'_2)| &\leq c2^{2j} \int\limits_{I_{j,m}} \left| \sum_{i=[I]+1}^{\infty} F_{k,i}(y) \right| dy \\ &\leq c2^{2j}|I_{j,m}|2^{-\nu_{[I]+1}s_k} \\ &\leq c2^{-j(s_k-1)}. \end{aligned}$$

That means, that condition (7.10) is fulfilled in this case.
D: $B_{x,2^{-K+2}}$ has a non-empty intersection with all Q_k for $k \geq k_0$.
Now $s_{K,x} = s_{k_0} - 1$ holds. For the estimate of $k^{\beta}_{j,m}(F')$ we have to distinguish between $\text{dist}(0, I_{j,m}) < 2^{-j}$ and $\text{dist}(0, I_{j,m}) \geq 2^{-j}$ again. For $\text{dist}(0, I_{j,m}) < 2^{-j}$ we follow the way of Case 1 and get

$$|k^{\beta}_{j,m}(F')| \leq c2^{2j} \int\limits_{I_{j,m}} \sum_{k=0}^{\infty}\sum_{i=1}^{\infty} 2^{-\nu_i s_k} \sum_{l}^{i} \omega(2^{\nu_i}y - l)dy.$$

By $\text{dist}(0, I_{j,m}) < 2^{-j}$ and the restrictions for the case D, it follows

$$
\begin{aligned}
|k^{\beta}_{j,m}(F')| &\leq c2^{2j} \sum_{k \gtrsim \frac{j-J}{2} \geq k_0} \sum_{\substack{i \in \mathbb{N} \\ \nu_i \gtrsim 2k}} 2^{-\nu_i s_{k_0}} \int_{Q_k} \sum_{l}^{i} \omega(2^{\nu_i} y - l) dy \\
&\leq c2^{2j} \sum_{k \gtrsim \frac{j-J}{2} \geq k_0} \sum_{\substack{i \in \mathbb{N} \\ \nu_i \gtrsim 2k}} 2^{-\nu_i s_{k_0}} 2^{-2k+\nu_i} 2^{-\nu_i} \\
&\leq c2^{2j} 2^{-j-js_{k_0}} \\
&\leq c2^{-j(s_{k_0}-1)}.
\end{aligned}
$$

For $\text{dist}(0, I_{j,m}) \geq 2^{-j}$ we use again the idea of Case 2 and find analogously to (7.5) and (7.6)

$$
\begin{aligned}
|k^{\beta}_{j,m}(F'_1)| &\leq c \sum_{k=k_0}^{j/2} \sum_{i=1}^{[I]} 2^{-\nu_i(s_k-1)} \int_{Q_k} \sum_{l}^{i} |\omega'(2^{\nu_i}(2^{-j}m + 2^{-j}y) - l)| dy \\
&\leq c \sum_{k=k_0}^{j/2} \sum_{i=1}^{[I]} 2^{-\nu_i(s_{k_0}-1)} 2^{-2k+\nu_i} 2^{-\nu_i} \\
&\leq c2^{-j(s_{k_0}-1)},
\end{aligned}
$$

in the same way as above. Moreover, similar to (7.7) and (7.8), we can estimate

$$
\begin{aligned}
|k^{\beta}_{j,m}(F'_2)| &\leq c2^{2j} \int_{I_{j,m}} \sum_{k=k_0}^{\infty} \left| \sum_{i=[I]+1}^{\infty} F_{k,i}(y) \right| dy \\
&\leq c2^{2j} |I_{j,m}| 2^{-\nu_{[I]+1} s_{k_0}} \\
&\leq c2^{-j(s_{k_0}-1)},
\end{aligned}
$$

so that condition (7.10) is fulfilled in this case.
<u>**E:**</u> $B_{x,2^{-K+2}}$ has a non-empty intersection with at most $Q_{k_a}, Q_{k_a+1}, \ldots, Q_{k_b}$. In this case we have $s_{K,x} = s_{k_a} - 1$. Furthermore, in (7.10), the supremum over x, K, in this case, can be estimated from above by the corresponding supremum in the case D with $k_a = k_0$. As in case D we find

$$
|k^{\beta}_{j,m}(F')| \leq c2^{-j(s_{k_a}-1)},
$$

which ensures, that condition (7.10) is fulfilled in this case.
Since we proved that (7.10) is fulfilled in all cases A-E, the proof is complete.

□

Remark 7.1 *Motivated by Theorem 16.2 in [24] we also wanted to prove that F' does not belong to any better space in terms of smoothness. Then we would have shown that F' is an extremal function in the space $B_{\infty}^{\mathbb{S},s_{\min}}(\mathbb{R})$. But we recently learned from [6] that the function constructed in the proof of Theorem 16.2 in [24] does not have the desired property, although the Theorem is correct, which is also shown in [6].*

The statement proved above is only a partial answer to the question under consideration, because the given function $\mathbb{S} : x \mapsto s(x)$ has a special dyadic structure. It is possible to generalize that a bit in the following way. At first we fill the gaps between the Q_k's. For the function

$$\widehat{F} = \sum_{k=0}^{\infty} \sum_{i=1}^{\infty} \widehat{F}_{k,i} \quad \text{with} \quad \widehat{F}_{k,i} = 2^{-\nu_i s_k} \sum_{l}^{i} \omega(2^{\nu_i} x - l),$$

where $\operatorname{supp} \widehat{F}_{k,i} \subset \widehat{Q}_k = [2^{-2k-2}, 2^{-2k-1})$ it follows by the previous calculations, that $\widehat{F}' \in B_{\infty}^{\widehat{\mathbb{S}},\widehat{s}_{\min}}(\mathbb{R})$, where $\widehat{\mathbb{S}} : x \mapsto \widehat{s}(x)$ is a lower semi-continuous function in $\mathbb{R}$ with

$$\widehat{s}(x) = \begin{cases} \widehat{s}_k - 1 & : \quad x \in \widehat{Q}_k \\ -\varepsilon & : \quad \text{otherwise} \end{cases}$$

for $\varepsilon > 0$ and a monotone increasing sequence $(\widehat{s}_k)_{k=0}^{\infty}$ with $0 < s_k < 1$ for all k. Now, by Corollary 3.2, we can state, that $F' + \widehat{F}' \in B_{\infty}^{\tilde{\mathbb{S}},\tilde{s}_{\min}}(\mathbb{R})$, where $\tilde{\mathbb{S}} : x \mapsto \tilde{s}(x)$ is a lower semi-continuous function in $\mathbb{R}$ with

$$\tilde{s}(x) = \begin{cases} s_k - 1 & : \quad x \in Q_k \\ \widehat{s}_k - 1 & : \quad x \in \widehat{Q}_k. \end{cases}$$

Now we only sketch the further way very roughly. We could repeat the whole procedure and divide every Q_k and $\widehat{Q}_k$ in subcubes by the same method. Instead of s_k and $\widehat{s}_k$ we would have suitable sequences $(s_{k,t})_{t=0}^{\infty}$ and $(\widehat{s}_{k,t})_{t=0}^{\infty}$ and could construct for every Q_k and $\widehat{Q}_k$ a function with the corresponding smoothness behavior again. If we repeat that over and over again, it seems possible, that in the limit, the dyadic structure of the step functions goes over to a continuous structure.

Remark 7.2 *One could also ask the inverse question: Given a function $f \in S'$, is it possible to construct a lower semi-continuous function $\mathbb{S} : x \mapsto s(x)$ in $\mathbb{R}^n$ with the properties*

$$f \in B_p^{\mathbb{S},s_0}(\mathbb{R}^n) \quad \text{and} \quad f \notin B_p^{\mathbb{S}+\varrho,s_0}(\mathbb{R}^n)$$

for every non-negative and non-vanishing lower semi-continuous function $\varrho(x)$? We had a few conjectures about how to construct such a function but did not succeed in proving one.

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